

CONSTRAINED SPLINE SMOOTHING*

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Abstract. Several results on constrained spline smoothing are obtained. In particular, we establish a general result, showing how one can constructively smooth any monotone or convex piecewise polynomial function (ppf) (or any q -monotone ppf, $q \geq 3$, with one additional degree of smoothness) to be of minimal defect while keeping it close to the original function in the \mathbb{L}_p - (quasi)norm. It is well known that approximating a function by ppfs of minimal defect (splines) avoids introduction of artifacts which may be unrelated to the original function; thus it is always preferable. On the other hand, it is usually easier to construct constrained ppfs with as few requirements on smoothness as possible. Our results allow us to obtain shape-preserving splines of minimal defect with equidistant or Chebyshev knots. The validity of the corresponding Jackson-type estimates for shape-preserving spline approximation is summarized; in particular, we show that the \mathbb{L}_p -estimates, $p \geq 1$, can be immediately derived from the \mathbb{L}_∞ -estimates.

Key words. splines, smoothing, minimal defect, moduli of smoothness, degree of approximation, Jackson-type estimates

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1. Introduction and the main results.

1.1. Notation. Let $\mathcal{S}_r(\mathbf{z}_n)$ be the space of all piecewise polynomial functions (ppf) of degree r (order $r + 1$) with the knots $\mathbf{z}_n := (z_i)_{i=0}^n$, $-1 := z_0 < z_1 < \dots < z_{n-1} < z_n := 1$. In other words, we say that $s \in \mathcal{S}_r(\mathbf{z}_n)$ if, on each interval (z_i, z_{i+1}) , $0 \leq i \leq n-1$, $s \in \Pi_r$, where Π_r denotes the space of algebraic polynomials of degree $\leq r$. Also, let $\tilde{\mathcal{S}}_r(\mathbf{z}_n) := \mathcal{S}_r(\mathbf{z}_n) \cap \mathbb{C}^{r-1}$ be the corresponding space of splines of minimal defect (highest smoothness).

As usual, $\mathbb{L}_p(J)$, $0 < p \leq \infty$, denotes the space of all measurable functions f on an interval J such that $\|f\|_{\mathbb{L}_p(J)} < \infty$, where $\|f\|_{\mathbb{L}_p(J)} := (\int_J |f(x)|^p dx)^{1/p}$ if $p < \infty$, and $\|f\|_{\mathbb{L}_\infty(J)} := \text{ess sup}_{x \in J} |f(x)|$. For $\mu \in \mathbb{N}$, the space of all μ -times continuously differentiable functions on J is denoted by $\mathbb{C}^\mu(J)$. Also, $\mathbb{C}(J)$ and $AC(J)$ denote the spaces of all continuous and locally absolutely continuous functions, respectively. (Note that if $f \in \mathbb{C}(J)$, then $\|f\|_{\mathbb{C}(J)} = \|f\|_{\mathbb{L}_\infty(J)}$.) The Sobolev space is defined by $\mathbb{W}_p^r(J) := \{f \in \mathbb{L}_p(J) \mid f^{(r-1)} \in AC(J) \text{ and } f^{(r)} \in \mathbb{L}_p(J)\}$.

For $k \in \mathbb{N}_0$, define

$$\Delta_h^k(f, x, J) := \begin{cases} \sum_{i=0}^k \binom{k}{i} (-1)^{k-i} f(x - kh/2 + ih) & \text{if } x \pm kh/2 \in J, \\ 0 & \text{otherwise.} \end{cases}$$

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The k th modulus of smoothness of $f \in \mathbb{L}_p(J)$ is defined by

$$\omega_k(f, t, J)_p := \sup_{0 < h \leq t} \left\| \Delta_h^k(f, \cdot, J) \right\|_{\mathbb{L}_p(J)},$$

and the Ditzian–Totik moduli of smoothness is

$$\omega_k^\varphi(f, t)_p := \sup_{0 < h \leq t} \left\| \Delta_{h\varphi(\cdot)}^k(f, \cdot) \right\|_{\mathbb{L}_p[-1,1]},$$

where $\varphi(x) := \sqrt{1 - x^2}$.

The set of q -monotone functions on J is denoted by $\Delta^q(J)$. Recall that $f \in \Delta^q(J)$ if the divided differences $[f; t_0, \dots, t_q]$ of order q of f are nonnegative for all choices of $(q + 1)$ distinct points t_0, \dots, t_q in J . If f is continuous, then $f \in \Delta^q(J)$ if $\Delta_h^q(f, x, J) \geq 0$ for all $x \in J$ and $h > 0$. It is well known that, for $q \geq 2$ and an open interval J , $f \in \Delta^q(J)$ if and only if $f^{(q-2)}$ exists and is convex on J .

The error of unconstrained approximation of f from a set U is denoted by

$$E(f, U)_p := \inf_{u \in U} \|f - u\|_{\mathbb{L}_p(J)},$$

and the error of q -monotone approximation (i.e., approximation of f by q -monotone elements of U) is

$$E^{(q)}(f, U)_p := E(f, U \cap \Delta^q(J))_p.$$

Throughout this paper, we also use the notation $\|f\|_p := \|f\|_{\mathbb{L}_p[-1,1]}$, $\Delta^q := \Delta^q[-1, 1]$, $|J| := \text{meas}(J)$, and

$$\omega_k(f, J)_p := \omega_k(f, |J|, J)_p \quad \text{and} \quad \omega_k(f, t)_p := \omega_k(f, t, [-1, 1])_p.$$

Given a partition $\mathbf{z}_n := (z_i)_{i=0}^n$, $-1 =: z_0 < z_1 < \dots < z_{n-1} < z_n =: 1$, we say that the partition $\tilde{\mathbf{z}}_m = (\tilde{z}_i)_{i=0}^m$ is a δ -remesh of \mathbf{z}_n if, for each $0 \leq j \leq n - 1$,

$$\max \{ \tilde{z}_{i+1} - \tilde{z}_i \mid [\tilde{z}_i, \tilde{z}_{i+1}] \cap (z_j, z_{j+1}) \neq \emptyset \} \leq \delta \min_{\nu=j-1, j, j+1} |z_{\nu+1} - z_\nu|$$

with z_{-1} and z_{n+1} defined to be (in this definition only!) $-\infty$ and $+\infty$, respectively. In other words, the largest interval $[\tilde{z}_i, \tilde{z}_{i+1}]$ intersecting (z_j, z_{j+1}) should have length at most δ times the length of $[z_j, z_{j+1}]$ or the lengths of (one or two) intervals adjacent to $[z_j, z_{j+1}]$, whichever is smaller. The class of all δ -remeshes of \mathbf{z}_n is denoted by $\mathcal{R}_\delta(\mathbf{z}_n)$.

Clearly, m , n , and δ are not independent. The smallest m such that there is $\tilde{\mathbf{z}}_m$ in $\mathcal{R}_\delta(\mathbf{z}_n)$ is determined not only by n and δ but also by the scale of the partition \mathbf{z}_n (see (1.1)).

It is well known that it is easier and less costly to obtain a good piecewise approximation to a function f than to assure, at the same time, the maximum smoothness of the approximating ppfs. On the other hand, replacing f by a ppf with minimal defect (spline) avoids introduction of artifacts which are unrelated to the original function, and may affect and complicate the problem one deals with. Thus, given two partitions \mathbf{z}_n and $\tilde{\mathbf{z}}_m$ with the only requirement that $\tilde{\mathbf{z}}_m$ is somewhat denser than \mathbf{z}_n (more precisely, $\tilde{\mathbf{z}}_m$ is an *arbitrary* δ -remesh of \mathbf{z}_n), we show in this paper how to smooth a general ppf on \mathbf{z}_n to a spline of maximum smoothness on $\tilde{\mathbf{z}}_m$, while staying close to the original ppf and preserving its shape characteristics. We prove that

such a $\delta > 0$ exists that splines on an arbitrary δ -remesh of \mathbf{z}_n may be constructed to satisfy the above requirements. In particular, as an illustration of these general results, we consider two special kinds of partitions of $[-1, 1]$ which are important in applications: the *uniform* partition $\mathbf{u}_n := (-1 + 2j/n)_{j=0}^n$ and the *Chebyshev* partition $\mathbf{t}_n := (-\cos(j\pi/n))_{j=0}^n$.

The following are some properties of classes $\mathcal{R}_\delta(\mathbf{z}_n)$.

- For any partition \mathbf{z}_n and $0 < \delta_1 \leq \delta_2$, $\mathcal{R}_{\delta_1}(\mathbf{z}_n) \subset \mathcal{R}_{\delta_2}(\mathbf{z}_n)$.
- If a partition \mathbf{z}_{kn}^* is a refinement of the partition \mathbf{z}_n obtained by subdividing each interval in \mathbf{z}_n into k equal subintervals, and $\tilde{\mathbf{z}}_m \in \mathcal{R}_\delta(\mathbf{z}_n)$, then $\tilde{\mathbf{z}}_m \in \mathcal{R}_{k\delta}(\mathbf{z}_{kn}^*)$.
- If $0 < \delta < 1$ and $\tilde{\mathbf{z}}_m \in \mathcal{R}_\delta(\mathbf{z}_n)$, then any interval in $\tilde{\mathbf{z}}_m$ is contained in the union of at most two intervals in \mathbf{z}_n .
- For any $m \geq n/\delta$, $\mathbf{u}_m \in \mathcal{R}_\delta(\mathbf{u}_n)$.
- For any $m \geq \max\{25/\delta, 1\}n$, $\mathbf{t}_m \in \mathcal{R}_\delta(\mathbf{t}_n)$.

1.2. Constrained smoothing. Let $\mathbf{z}_n := \{z_0, \dots, z_n \mid -1 =: z_0 < z_1 < \dots < z_n := 1\}$ be a partition of $[-1, 1]$, and extend the notation by setting $z_j := z_0$, $j < 0$, and $z_j := z_n$, $j > n$. We denote the scale of the partition \mathbf{z}_n by

$$(1.1) \quad \vartheta(\mathbf{z}_n) := \max_{0 \leq j \leq n-1} \frac{|J_{j\pm 1}|}{|J_j|},$$

where $J_j := [z_j, z_{j+1}]$. We also denote $J_j := [(z_{j-1} + z_j)/2, (z_j + z_{j+1})/2]$.

The following theorem is our main result on constrained spline smoothing.

THEOREM 1.1. *Let $q \in \mathbb{N}$, $r \in \mathbb{N}$, and let $\mathbf{z}_n = (z_i)_{i=0}^n$ be a partition of $[-1, 1]$. There is a constant $\delta = \delta(q, r)$ such that for each $s \in \mathcal{S}_{q+r}(\mathbf{z}_n) \cap \Delta^q$ such that*

$$(1.2) \quad s \in \mathbb{C}^{q-1}[-1, 1],$$

and any $\tilde{\mathbf{z}}_m \in \mathcal{R}_\delta(\mathbf{z}_n)$ (i.e., $\tilde{\mathbf{z}}_m$ is a δ -remesh of \mathbf{z}_n), there exists a spline $\tilde{s} \in \tilde{\mathcal{S}}_{q+r}(\tilde{\mathbf{z}}_m) \cap \Delta^q$ satisfying

$$\|s - \tilde{s}\|_{\mathbb{L}_p(J_j)} \leq c(p, q, r)\omega_{q+r+1}(s, J_j)_p, \quad 0 \leq j \leq n,$$

for all $0 < p \leq \infty$. Moreover, the construction of \tilde{s} does not depend on p .

The proof of Theorem 1.1 (as well as the proof of Theorem 1.2 below) is postponed until section 3.

Recall that, for any $q \geq 2$, $\Delta^q \subset \mathbb{C}^{q-2}(-1, 1)$; i.e., any q -monotone function is in \mathbb{C}^{q-2} . Hence, the smoothness provided by shape itself does not guarantee the applicability of the above result; one needs to assume/gain one additional smoothness degree.

It turns out that for $q = 1$ and $q = 2$ the gain of this additional degree of smoothness is not difficult (see section 3.2), and so we get the following stronger result in the case for $q \leq 2$.

THEOREM 1.2. *Let $q = 1$ or 2 , $r \in \mathbb{N}$, and let $\mathbf{z}_n = (z_i)_{i=0}^n$ be a partition of $[-1, 1]$. There is a constant $\delta = \delta(r)$ such that for each $s \in \mathcal{S}_{q+r}(\mathbf{z}_n) \cap \Delta^q$ and any $\tilde{\mathbf{z}}_m \in \mathcal{R}_\delta(\mathbf{z}_n)$ there exists a spline $\tilde{s} \in \tilde{\mathcal{S}}_{q+r}(\tilde{\mathbf{z}}_m) \cap \Delta^q$ satisfying*

$$\|s - \tilde{s}\|_{\mathbb{L}_p(J_j)} \leq c(p, r)\omega_{q+r+1}(s, J_j)_p, \quad 0 \leq j \leq n,$$

for all $0 < p \leq \infty$.

Note that an analog of Theorem 1.2 is also valid (and is actually simpler) in the case $r = 0$ (see Lemma 3.8 and Corollary 3.9).

In case the partitions \mathbf{z}_n and $\tilde{\mathbf{z}}_m$ are either both uniform or both Chebyshev, Theorem 1.1 can be restated as follows.

COROLLARY 1.3. *Let $q \in \mathbb{N}$, $r \in \mathbb{N}$, and let $\mathbf{z}_n = (z_i)_{i=0}^n$ denote either \mathbf{u}_n or \mathbf{t}_n . There is a constant $m_0 = m_0(q, r)$ such that for each $s \in \mathcal{S}_{q+r}(\mathbf{z}_n) \cap \Delta^q \cap \mathbb{C}^{q-1}[-1, 1]$, and any $m \geq m_0 n$, there exists a spline $\tilde{s} \in \tilde{\mathcal{S}}_{q+r}(\mathbf{z}_m) \cap \Delta^q$, where \mathbf{z}_m is either \mathbf{u}_m or \mathbf{t}_m , respectively, satisfying*

$$\|s - \tilde{s}\|_{\mathbb{L}_p(J_j)} \leq c(p, q, r)\omega_{q+r+1}(s, J_j)_p, \quad 0 \leq j \leq n,$$

for all $0 < p \leq \infty$.

We remark that in view of Theorem 1.2, in the case $q = 1$ and $q = 2$, the condition that s is in $\mathbb{C}^{q-1}[-1, 1]$ in Corollary 1.3 can be removed.

Finally, we remark that it is still an open question whether the condition (1.2) in the statement of Theorem 1.1 can be removed if $q \geq 3$.

2. Applications: Jackson-type estimates.

2.1. Monotone and convex spline approximation: $p = \infty$. The following theorem is rather well known. Its positive part follows from Whitney’s inequality ($q = 1$ and $1 \leq k + \nu \leq 2$), [11, Lemma 2] ($q = 1$, $\nu \geq 1$), [13, Corollary 2.4] ($q = 2$, $\nu \geq 2$), and [6] ($q = 2$, $2 \leq k + \nu \leq 3$). The negative part follows from [17] and [15, p. 141].

THEOREM 2.1 ($p = \infty$). *Let $q = 1$ or 2 , and $k, \nu \in \mathbb{N}_0$ be such that either $\nu \geq q$ or $q \leq k + \nu \leq q + 1$ (see Figures 1 and 2). Then, for every $f \in \Delta^q \cap \mathbb{C}^\nu[-1, 1]$, $n \in \mathbb{N}$, and any partition $\mathbf{z}_n = (z_i)_{i=0}^n$ of $[-1, 1]$, there exists $s \in \mathcal{S}_{k+\nu-1}(\mathbf{z}_n) \cap \Delta^q$ such that*

$$\|f - s\|_{\mathbb{L}_\infty(J_j)} \leq c(k, \nu, \vartheta(\mathbf{z}_n))|J_j|^\nu \omega_k(f^{(\nu)}, J_j)_\infty, \quad 0 \leq j \leq n - 1.$$

Moreover, this estimate is no longer true in general for k and ν which do not satisfy the above conditions. (This means that for each partition \mathbf{z}_n and any constant $c(k, \nu, \vartheta(\mathbf{z}_n))$, there exists a function $f \in \Delta^q \cap \mathbb{C}^\nu[-1, 1]$, such that the above estimate is invalid for any $s \in \mathcal{S}_{k+\nu-1}(\mathbf{z}_n) \cap \Delta^q$.)

We note that the case $q = 2$, $k + \nu = 1$ is excluded from the statement of Theorem 2.1 (as well as the statements of Theorem 2.4 and Corollaries 2.2, 2.3, 2.5, and 2.6). Indeed, the only convex piecewise constant ppfs are constant functions on $[-1, 1]$.

Using Theorem 1.2 we can now obtain the following consequence of this result for monotone and convex approximation by splines of any smoothness.

COROLLARY 2.2 ($p = \infty$). *Let $q = 1$ or 2 , and $k, \nu \in \mathbb{N}_0$ be such that either $\nu \geq q$ or $q \leq k + \nu \leq q + 1$. Then, for any $r \geq k + \nu - 1$ and partition $\mathbf{z}_n = (z_i)_{i=0}^n$ of $[-1, 1]$, there exists a constant $\delta = \delta(r)$ such that for any $\tilde{\mathbf{z}}_m \in \mathcal{R}_\delta(\mathbf{z}_n)$ and every $f \in \Delta^q \cap \mathbb{C}^\nu[-1, 1]$, there exists a spline $\tilde{s} \in \tilde{\mathcal{S}}_r(\tilde{\mathbf{z}}_m) \cap \Delta^q$ satisfying*

$$\|f - \tilde{s}\|_{\mathbb{L}_\infty(J_j)} \leq c(k, \nu, r, \vartheta(\mathbf{z}_n))|J_j|^\nu \omega_k(f^{(\nu)}, [z_{j-1}, z_{j+1}])_\infty$$

for all $0 \leq j \leq n$. Moreover, this estimate is no longer true in general for k and ν which do not satisfy the above conditions.

Taking into account Corollary 1.3 and the fact that $|J_j| = n^{-1}$ (if $\mathbf{z}_n = \mathbf{u}_n$) and $|J_j| \sim \varphi(x)n^{-1} + n^{-2}$, $x \in J_j$ (if $\mathbf{z}_n = \mathbf{t}_n$), this, in turn, immediately implies the following result.

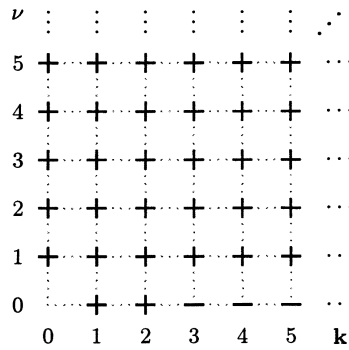


FIG. 1. $q = 1, p = \infty$.

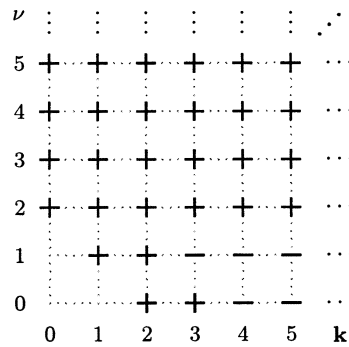


FIG. 2. $q = 2, p = \infty$.

COROLLARY 2.3. *Let $q = 1$ or 2 , and $k, \nu \in \mathbb{N}_0$ be such that either $\nu \geq q$ or $q \leq k + \nu \leq q + 1$. Then, for every $f \in \Delta^q \cap \mathbb{C}^\nu[-1, 1]$, $n \in \mathbb{N}$, and $r \geq k + \nu - 1$, we have*

$$(2.1) \quad E^{(q)}(f, \tilde{\mathcal{S}}_r(\mathbf{u}_n))_\infty \leq c(k, \nu, r)n^{-\nu}\omega_k(f^{(\nu)}, n^{-1})_\infty$$

and

$$(2.2) \quad E^{(q)}(f, \tilde{\mathcal{S}}_r(\mathbf{t}_n))_\infty \leq c(k, \nu, r)n^{-\nu}\omega_k^{\varphi}(f^{(\nu)}, n^{-1})_\infty.$$

Moreover, these estimates are no longer true in general for k and ν which do not satisfy the above conditions.

For the reader's convenience we describe the above results using arrays in Figures 1 and 2. In these figures as well as Figures 3 and 4 in the case $1 \leq p < \infty$ (with obvious modifications), the symbols “-” and “+” have the following meaning.

- The symbol “+” in the position (k, ν) means that inequalities (2.1) and (2.2) are valid for all $f \in \Delta^q \cap \mathbb{C}^\nu[-1, 1]$.
- The symbol “-” in the position (k, ν) means that inequalities (2.1) and (2.2) are not true in general, i.e., there are functions $f \in \Delta^q \cap \mathbb{C}^\nu[-1, 1]$ for which these inequalities fail.

We remark that the estimates in Corollary 2.3 (and the arrays in Figures 1 and 2) are well known for algebraic polynomials as well as splines of low smoothness. For splines of minimal defect, only some special cases when $\mathbf{z}_n = \mathbf{u}_n$ were known (see, e.g., [3, Theorem 1], [10, Theorem 2.1], [2, Theorem 1], [8, Corollaries 5.3 and 5.4]).

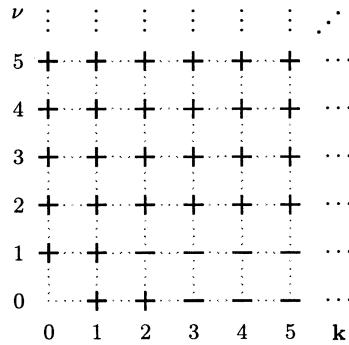


FIG. 3. $q = 1, 1 \leq p < \infty$.

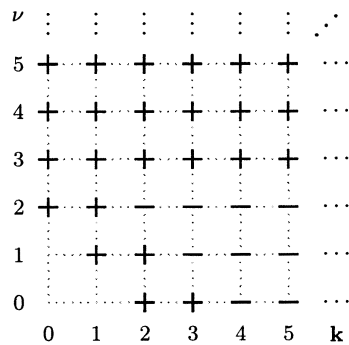


FIG. 4. $q = 2, 1 \leq p < \infty$.

2.2. Monotone and convex spline approximation: $1 \leq p < \infty$.

THEOREM 2.4 ($1 \leq p < \infty$). *Let $q = 1$ or $2, 1 \leq p < \infty$, and $k, \nu \in \mathbb{N}_0$ be such that either $\nu \geq q + 1$ or $q \leq k + \nu \leq q + 1$. Then, for every $f \in \Delta^q \cap \mathbb{W}_p^\nu[-1, 1], n \in \mathbb{N}$, and any partition $\mathbf{z}_n = (z_i)_{i=0}^n$ of $[-1, 1]$, there exists $s \in \mathcal{S}_{k+\nu-1}(\mathbf{z}_n) \cap \Delta^q$ such that*

$$\|f - s\|_{L_p(J_j)} \leq c(k, \nu, \vartheta(\mathbf{z}_n)) |J_j|^\nu \omega_k(f^{(\nu)}, J_j)_p, \quad 0 \leq j \leq n - 1.$$

Moreover, this estimate is no longer true in general for k and ν which do not satisfy the above conditions.

The positive part of Theorem 2.4 for $\nu \geq 1$ follows from Theorem 2.1 and the well-known inequality

$$\omega_{r+1}(f, J)_\infty \leq c|J|^{1-1/p} \omega_r(f', J)_p,$$

where J is a closed interval, $f \in \mathbb{W}_p^1(J), 1 \leq p < \infty$, and $r \in \mathbb{N}_0$. For $\nu = 0$, the case $(q, k) = (1, 1)$ is straightforward, $(q, k) = (2, 3)$ is shown in [4, Theorem 1.2], and the cases $(q, k) = (1, 2)$ and $(2, 2)$ are established following the proof of [4, Theorem 1.2] using piecewise linear ppfs which interpolate f at the knots instead of piecewise quadratic functions in [4, Lemma 2.2]. The negative part of Theorem 2.4 is a consequence of [7, Theorem 1].

As in the case $p = \infty$, Theorem 1.2 yields the following stronger result for monotone and convex approximation by splines of any smoothness.

COROLLARY 2.5 ($1 \leq p < \infty$). *Let $q = 1$ or $2, 1 \leq p < \infty$, and $k, \nu \in \mathbb{N}_0$ be such that either $\nu \geq q + 1$ or $q \leq k + \nu \leq q + 1$. Then, for any $r \geq k + \nu - 1$ and*

any partition $\mathbf{z}_n = (z_i)_{i=0}^n$ of $[-1, 1]$, there exists a constant $\delta = \delta(r)$ such that for any $\tilde{\mathbf{z}}_m \in \mathcal{R}_\delta(\mathbf{z}_n)$ and every $f \in \Delta^q \cap \mathbb{W}_p^\nu[-1, 1]$ there exists a spline $\tilde{s} \in \tilde{\mathcal{S}}_r(\tilde{\mathbf{z}}_m) \cap \Delta^q$ satisfying

$$\|f - \tilde{s}\|_{\mathbb{L}_p(J_j)} \leq c(k, \nu, r, \vartheta(\mathbf{z}_n)) |J_j|^\nu \omega_k(f^{(\nu)}, [z_{j-1}, z_{j+1}])_p$$

for all $0 \leq j \leq n$. Moreover, this estimate is no longer true in general for k and ν which do not satisfy the above conditions.

Recalling that for $f \in \mathbb{L}_p[-1, 1]$, $1 \leq p < \infty$, and $k, \mu \in \mathbb{N}$ the following estimates are true (see, e.g., [14], [5]):

$$\sum_{j=0}^{n-\mu-1} \omega_k \left(f, \cup_{i=j}^{j+\mu} J_i \right)_p^p \leq \begin{cases} c(k, \mu) \omega_k(f, n^{-1})_p^p & \text{if } \mathbf{z}_n = \mathbf{u}_n, \\ c(k, \mu) \omega_k^\varphi(f, n^{-1})_p^p & \text{if } \mathbf{z}_n = \mathbf{t}_n, \end{cases}$$

we get the following consequence of Corollary 2.5.

COROLLARY 2.6. *Let $q = 1$ or 2 , $1 \leq p < \infty$, and $k, \nu \in \mathbb{N}_0$ be such that either $\nu \geq q + 1$ or $q \leq k + \nu \leq q + 1$. Then, for any $f \in \Delta^q \cap \mathbb{W}_p^\nu[-1, 1]$, $n \in \mathbb{N}$, and $r \geq k + \nu - 1$, we have*

$$(2.3) \quad E^{(q)}(f, \tilde{\mathcal{S}}_r(\mathbf{u}_n))_p \leq c(k, \nu, r) n^{-\nu} \omega_k(f^{(\nu)}, n^{-1})_p$$

and

$$(2.4) \quad E^{(q)}(f, \tilde{\mathcal{S}}_r(\mathbf{t}_n))_p \leq c(k, \nu, r) n^{-\nu} \omega_k^\varphi(f^{(\nu)}, n^{-1})_p.$$

Moreover, these estimates are no longer true in general for k and ν which do not satisfy the above conditions.

3. Further results and proofs. Everywhere in this section, if p is a polynomial piece of a spline s on an interval J , i.e., $p := s|_J$, then $p(x)$ for $x \notin J$ is the polynomial extension of $s|_J$.

3.1. Constrained spline smoothing: Proof of Theorem 1.1.

LEMMA 3.1 (see [9, Lemma 2.4 and Corollary 2.5]). *Let $s \in \mathcal{S}_r(\mathbf{z}_n)$, and suppose that the polynomials $p_j := s|_{[z_j, z_{j+1}]}$, $0 \leq j \leq n - 1$. Then, for every $1 \leq j \leq n - 1$ and all $0 < p \leq \infty$,*

$$|J_j|^{k+1/p} |p_j^{(k)}(z_j) - p_{j-1}^{(k)}(z_j)| \leq c(p, r, \vartheta(\mathbf{z}_n)) \omega_{r+1}(s, [z_{j-1}, z_{j+1}])_p$$

for $0 \leq k \leq r$, and

$$\|p_j - p_{j-1}\|_{\mathbb{L}_p[z_{j-1}, z_{j+1}]} \leq c(p, r, \vartheta(\mathbf{z}_n)) \omega_{r+1}(s, [z_{j-1}, z_{j+1}])_p.$$

LEMMA 3.2 (Markov’s inequality). *For any polynomial $\mathbf{p} \in \Pi_r$,*

$$\|\mathbf{p}'\|_{\mathbb{C}[a,b]} \leq \frac{2r^2}{b-a} \|\mathbf{p}\|_{\mathbb{C}[a,b]}.$$

LEMMA 3.3. *Let $r \in \mathbb{N}$, $S \in \mathcal{S}_r(\{-1, 0, 1\})$ (i.e., S is a ppf of degree $\leq r$ on $[-1, 1]$ with the only breakpoint at 0), and let p_1 and p_2 be the polynomial pieces of S : $p_1 := S|_{[-1,0]}$, $p_2 := S|_{[0,1]}$. If S is nonnegative on $[-1, 1]$, then there exists an interval I , $I \subset [-1, 0]$ or $I \subset [0, 1]$ such that $|I| \geq \frac{1}{4r^2}$ and*

$$S(x) \geq c_1(r) \|p_1 - p_2\|_\infty \quad \text{for all } x \in I.$$

Proof. Let $p \in \Pi_r$ be nonnegative on $[0, 1]$ (the same argument applies for $[-1, 0]$). Then, there exists an interval $I \subset [0, 1]$, $|I| \geq \frac{1}{4r^2}$ such that

$$p(x) \geq \frac{1}{2} \|p\|_{\mathbb{C}[0,1]} \quad \text{for all } x \in I.$$

Indeed, let $x^* \in [0, 1]$ be a point satisfying $p(x^*) = \|p\|_{\mathbb{C}[0,1]}$. By Lemma 3.2, $\|p'\|_{\mathbb{C}[0,1]} \leq 2r^2 \|p\|_{\mathbb{C}[0,1]}$, and so

$$p(x) \geq p(x^*) - 2r^2 \|p\|_{\mathbb{C}[0,1]} |x - x^*|, \quad x \in [0, 1].$$

Hence, if $x \in [0, 1]$ and $|x - x^*| \leq \frac{1}{4r^2}$, then $p(x) \geq \frac{1}{2} \|p\|_{\mathbb{C}[0,1]}$, as required.

Now suppose, without loss of generality, that $\|p_2\|_\infty \geq \|p_1\|_\infty$. Then

$$\|p_1 - p_2\|_\infty \leq \|p_1\|_\infty + \|p_2\|_\infty \leq 2 \|p_2\|_\infty \leq c(r) \|p_2\|_{\mathbb{C}[0,1]},$$

and the above observation can be used with $p := p_2$. □

The following lemma follows from Beatson [1, Lemma 3.2].

LEMMA 3.4. *Let $r \in \mathbb{N}$, $d := 2r^2$, and $\mathbf{p}_1, \mathbf{p}_2 \in \Pi_r$. For any knot sequence $\mathbf{x}_d := (x_i)_{i=0}^d$, $a = x_0 < x_1 < \dots < x_d = b$, there exists a spline $s \in \tilde{\mathcal{S}}_r(\mathbf{x}_d)$ such that*

- (i) $s(x)$ is a number between $\mathbf{p}_1(x)$ and $\mathbf{p}_2(x)$ for $x \in [a, b]$,
- (ii) $s \equiv \mathbf{p}_1$ on $(-\infty, a]$, and $s \equiv \mathbf{p}_2$ on $[b, \infty)$.

LEMMA 3.5. *Let $r \in \mathbb{N}$, $S \in \mathcal{S}_r(\{-1, 0, 1\})$ (i.e., S is a ppf of degree $\leq r$ on $[-1, 1]$ with the only breakpoint at 0), and assume that S is nonnegative on $[-1, 1]$. Suppose that $d := 2r^2$, and a knot sequence $\mathbf{z}_{2d+1} := (z_i)_{i=0}^{2d+1}$, $-1 < z_0 < z_1 < \dots < z_{2d+1} < 1$ is such that $z_d \leq 0 \leq z_{d+1}$ (in other words, there are $(d + 1)$ knots to the left and to the right of zero). Then, there exists $\tilde{S} \in \tilde{\mathcal{S}}_r(\mathbf{z}_{2d+1})$ satisfying*

- (i) $\tilde{S} \geq 0$ on $[-1, 1]$,
- (ii) $\tilde{S}(x) = S(x)$ for all $x \notin [z_0, z_{2d+1}]$,
- (iii) $\|S - \tilde{S}\|_\infty \leq 2 \|p_1 - p_2\|_\infty$, where $p_1 := S|_{[-1,0]}$, $p_2 := S|_{(0,1]}$ are the polynomial pieces of S .

Proof. Let $\tilde{p}(x) := p_1(x) + \|p_1 - p_2\|_\infty$. Note that $\tilde{p}(x) \geq p_1(x) \geq 0$ for $x \in [-1, 0]$, and $\tilde{p}(x) \geq p_2(x) \geq 0$ for $x \in [0, 1]$.

We now use Lemma 3.4 twice. First, we “glue” p_1 with \tilde{p} using Lemma 3.4 with $\mathbf{p}_1 = p_1$, $\mathbf{p}_2 = \tilde{p}$, $\mathbf{x}_d = (z_i)_{i=0}^d$, and $[a, b] = [z_0, z_d]$. Second, we “glue” \tilde{p} with p_2 using Lemma 3.4 with $\mathbf{p}_1 = \tilde{p}$, $\mathbf{p}_2 = p_2$, $\mathbf{x}_d = (z_i)_{i=d+1}^{2d+1}$, and $[a, b] = [z_{d+1}, z_{2d+1}]$. As a result, we get a spline $\tilde{S} \in \tilde{\mathcal{S}}_r(\mathbf{z}_{2d+1})$ such that $\tilde{p} \geq \tilde{S} \geq p_1 \geq 0$ on $[z_0, z_d]$, $\tilde{p} \geq \tilde{S} \geq p_2 \geq 0$ on $[z_{d+1}, z_{2d+1}]$, $\tilde{S}(x) = S(x) \geq 0$ for $x \notin [z_0, z_{2d+1}]$, and $\tilde{S}(x) = \tilde{p}(x) \geq 0$ for $x \in [z_d, z_{d+1}]$. Clearly,

$$\|S - \tilde{S}\|_\infty \leq \max_{i=1,2} \|\tilde{p} - p_i\|_\infty \leq 2 \|p_1 - p_2\|_\infty,$$

and the proof is complete. □

The following lemma is the main tool for constrained smoothing.

LEMMA 3.6. *Let $q \in \mathbb{N}$, $r \in \mathbb{N}$. There exists a constant $\delta(q, r) > 0$ such that, for each ppf $s \in \Delta^q \cap \mathbb{C}^{q-1}[-1, 1]$ of degree $\leq q + r$ with the only knot at 0 (i.e., $s \in \mathcal{S}_{q+r}(\{-1, 0, 1\})$), and each partition $\mathbf{z}_l = (z_i)_{i=0}^l$, $-1 = z_0 < z_1 < \dots < z_{l-1} < z_l = 1$ with*

$$\max_{0 \leq i \leq l-1} (z_{i+1} - z_i) =: \delta < \delta(q, r),$$

there exists a spline $\tilde{s} \in \Delta^q \cap \tilde{\mathfrak{S}}_{q+r}(\mathbf{z}_l)$ (i.e., \tilde{s} is a spline of minimal defect) such that $\tilde{s} \equiv s$ in neighborhoods of -1 and 1 , and

$$(3.1) \quad \|s - \tilde{s}\|_p \leq c(p, q, r)\omega_{q+r+1}(s, 1)_p$$

for all $0 < p \leq \infty$.

Proof. Since $s^{(q-1)}$ is absolutely continuous and differentiable everywhere except for 0 (because it is a continuous ppf of degree $\leq r + 1$ with the only breakpoint at 0), we conclude that $S := s^{(q)}$ is a nonnegative ppf of degree $\leq r$ with the only knot at 0 . Let $p_1 := S|_{[-1,0)}$, $p_2 := S|_{(0,1]}$ be the polynomial pieces of S , and denote $\eta := \|p_1 - p_2\|_\infty$ and $d := 2r^2$. Lemma 3.3 implies that there exists an interval I (without loss of generality, we may assume that it is a subset of $[0, 1]$) such that $|I| \geq (2d)^{-1}$ and

$$(3.2) \quad S(x) \geq c_1\eta \quad \text{for all } x \in I,$$

where c_1 depends on r only.

If $P_1 := s|_{[-1,0)}$, $P_2 := s|_{(0,1]}$ are the original polynomial pieces of s , then $p_1 = P_1^{(q)}$, $p_2 = P_2^{(q)}$, and using Lemma 3.2 as well as Lemma 3.1 we have

$$\begin{aligned} \eta &= \left\| P_1^{(q)} - P_2^{(q)} \right\|_\infty \leq (q+r)^{2q} \|P_1 - P_2\|_\infty \\ &\leq c(p, q, r) \|P_1 - P_2\|_p \leq c(p, q, r)\omega_{q+r+1}(s, 1)_p. \end{aligned}$$

Now, let I' be the subinterval of I with the same right endpoint as I and length $|I'| = (5d)^{-1}$, so that

$$I' \cap [-(5d)^{-1}, (5d)^{-1}] = \emptyset.$$

Suppose now that the index i^* is such that $z_{i^*} \leq 0 < z_{i^*+1}$, and that $\delta < (5d(d+2))^{-1}$. Then each of the intervals $[-(5d)^{-1}, 0]$ and $[0, (5d)^{-1}]$ contains at least $(d+1)$ points from \mathbf{z}_l :

$$(z_i)_{i=i^*-d}^{i^*} \subset [-(5d)^{-1}, 0] \quad \text{and} \quad (z_i)_{i=i^*+1}^{i^*+d+1} \subset [0, (5d)^{-1}].$$

We can apply Lemma 3.5 to construct a nonnegative spline \tilde{S} of degree r and highest smoothness ($\tilde{S} \in \mathbb{C}^{r-1}$) having knots $(z_i)_{i=i^*-d}^{i^*+d+1}$ only, coinciding with S outside $J_\star := [z_{i^*-d}, z_{i^*+d+1}]$, and such that

$$\|S - \tilde{S}\|_\infty \leq 2\eta.$$

We define s_\star to be such that $s_\star^{(q)} \equiv \tilde{S}$ and $s_\star^{(\nu)}(-1) = s^{(\nu)}(-1)$ for all $0 \leq \nu \leq q-1$. Therefore,

$$s_\star(x) := \sum_{\nu=0}^{q-1} \frac{s^{(\nu)}(-1)}{\nu!} (x+1)^\nu + \frac{1}{(q-1)!} \int_{-1}^x (x-t)^{q-1} \tilde{S}(t) dt.$$

The spline s_\star is almost what we need. Namely, it is in $\Delta^q \cap \tilde{\mathfrak{S}}_{q+r}(\mathbf{z}_l)$, it coincides with s in a neighborhood of -1 , and since S and \tilde{S} may differ only on an interval J_\star of length $|J_\star| \leq (2d+1)\delta$, we have

$$\begin{aligned} \|s - s_\star\|_\infty &\leq \left\| \frac{1}{(q-1)!} \int_{-1}^x (x-t)^{q-1} (S(t) - \tilde{S}(t)) dt \right\|_\infty \\ &\leq \frac{2^q \eta}{(q-1)!} |J_\star| \leq 2^q (2d+1) \eta \delta =: c_2 \eta \delta. \end{aligned}$$

The only property that s_* does not have is that it may not be coinciding with s in a neighborhood of 1. To remedy this, we use Lemma 3.4 and “glue” s_* to s on the interval I' preserving its properties.

Let $[a, b] := I'$ and recall that $b - a = (5d)^{-1}$. Now, let $d_1 := 2(q + r)^2$ and take $\delta < h := \frac{b-a}{2(d_1+1)} = \frac{1}{10d(d_1+1)}$. Then, each interval $[a + jh, a + (j+1)h]$, $0 \leq j \leq 2d_1 + 1$, contains at least one point z_{i_j} from \mathbf{z}_l , and we set $\mathbf{x}_{d_1} = (x_j)_{j=0}^{d_1}$, where $x_j := z_{i_{(2j)}}$, $0 \leq j \leq d_1$. Note that

$$\max_{0 \leq j \leq d_1} (x_{j+1} - x_j) \leq 3h \quad \text{and} \quad \min_{0 \leq j \leq d_1} (x_{j+1} - x_j) \geq h.$$

Taking into account that both s and s_* are polynomials of degree $\leq q + r$ on I' , Lemma 3.4 implies that there exists a spline $\tilde{s} \in \tilde{\mathcal{S}}_{q+r}(\mathbf{z}_l)$ such that the only knots of \tilde{s} inside I' are x_j , $0 \leq j \leq d_1$, $\tilde{s}(x)$ is a number between $s(x)$ and $s_*(x)$ for $x \in [x_0, x_{d_1}] \subset I'$, and \tilde{s} coincides with s_* on $[-1, x_0]$ and with s on $[x_{d_1}, 1]$. Hence,

$$\|s - \tilde{s}\|_\infty \leq \|s - s_*\|_\infty \leq c_2\eta\delta$$

(so that (3.1) immediately follows), and it only remains to verify that $\tilde{s} \in \Delta^q[x_0, x_{d_1}]$. To this end, it suffices to show that $\tilde{s}^{(q)}(x) \geq 0$ for all $x \in J_j := [x_j, x_{j+1}]$, $0 \leq j \leq d_1 - 1$. Let such an interval J_j be fixed. Using Lemma 3.2 and the fact that $\tilde{s} - s$ is a polynomial of degree $\leq q + r$ on J_j , we have

$$\begin{aligned} \left\| \tilde{s}^{(q)} - s^{(q)} \right\|_{\mathbb{C}(J_j)} &\leq \left(\frac{2(q+r)^2}{|J_j|} \right)^q \|s - \tilde{s}\|_{\mathbb{C}(J_j)} \leq \left(\frac{2(q+r)^2}{h} \right)^q c_2\eta\delta \\ &= (10dd_1(d_1+1))^q c_2\eta\delta =: c_3\eta\delta, \end{aligned}$$

where c_3 depends only on r and q . Therefore, for every $x \in J_j$, using (3.2) we have

$$\tilde{s}^{(q)}(x) \geq s^{(q)}(x) - c_3\eta\delta \geq c_1\eta - c_3\eta\delta \geq 0,$$

provided $\delta \leq c_1/c_3$. Combining the above restrictions on δ we see that it is possible to take

$$\delta(q, r) := \min \{ (10d(d_1+1))^{-1}, c_1/c_3 \},$$

and the construction of \tilde{s} is complete. \square

COROLLARY 3.7. *Let $q \in \mathbb{N}$, $r \in \mathbb{N}$. Then there exists a constant $\delta(q, r) > 0$ such that, for each ppf $s \in \Delta^q \cap \mathbb{C}^{q-1}[a, b]$ of degree $\leq q + r$ with the only knot at $c \in (a, b)$ (i.e., $s \in \mathcal{S}_{q+r}(\{a, c, b\})$), and each partition $\mathbf{z}_l = (z_i)_{i=0}^l$, $a = z_0 < z_1 < \dots < z_{l-1} < z_l = b$ with*

$$\max_{0 \leq i \leq l-1} (z_{i+1} - z_i) < \delta(q, r) \min\{b - c, c - a\},$$

there exists a spline $\tilde{s} \in \Delta^q \cap \tilde{\mathcal{S}}_{q+r}(\mathbf{z}_l)$ (i.e., \tilde{s} is a spline of minimal defect) such that $\tilde{s} \equiv s$ in neighborhoods of a and b , and

$$\|s - \tilde{s}\|_{\mathbb{L}_p[a,b]} \leq c(p, q, r)\omega_{q+r+1}(s, [a, b])_p$$

for all $0 < p \leq \infty$.

Proof. Without loss of generality, assume $c \geq (b + a)/2$. We apply Lemma 3.6 to $s^*(x) := s(c + (b - c)x)$ and those knots $(z_i - c)/(b - c)$ which belong to $[-1, 1]$. Now, extending the resulting ppf \tilde{s}^* polynomially from $[-1, 1]$ to all \mathbb{R} , we can go back to the original interval $[a, b]$ using $\tilde{s}(t) := \tilde{s}^*((t - c)/(b - c))$. \square

Proof of Theorem 1.1. It is sufficient to apply Corollary 3.7 with $[a, b] = J_j$ and $c = z_j$, for each $1 \leq j \leq n - 1$. Construction of \tilde{s} is now obvious. \square

3.2. Convex and monotone spline smoothing: Auxiliary results for the proof of Theorem 1.2. Recall that for a partition $\mathbf{z}_n := (z_i)_{i=0}^n$, $a := z_0 < z_1 < \dots < z_{n-1} < z_n := b$ of an interval $[a, b]$, $z_j := a$, $j < 0$, and $z_j := b$, $j > n$, and that $J_j = [z_j, z_{j+1}]$. The following lemma shows how a piecewise polynomial convex function can be smoothed to be continuously differentiable without adding any extra knots and keeping the error small.

LEMMA 3.8 (convex smoothing). *Let $r \in \mathbb{N}_0$, $\mathbf{z}_n := (z_i)_{i=0}^n$, $a := z_0 < z_1 < \dots < z_{n-1} < z_n := b$ be a partition of $[a, b]$, and let $s \in \Delta^2 \cap \mathcal{S}_{r+2}(\mathbf{z}_n)$. Then, there exists $\tilde{s} \in \Delta^2 \cap \mathcal{S}_{r+2}(\mathbf{z}_n) \cap \mathcal{C}^1[a, b]$ such that, for any $1 \leq j \leq n - 1$ and all $0 < p \leq \infty$,*

$$(3.3) \quad \|s - \tilde{s}\|_{\mathbb{L}_p[z_{j-1}, z_{j+1}]} \leq c(p, r, \vartheta(\mathbf{z}_n)) \omega_{r+3}(s, [z_{j-2}, z_{j+2}])_p$$

and

$$(3.4) \quad \|s' - \tilde{s}'\|_{\mathbb{L}_p[z_{j-1}, z_{j+1}]} \leq c(p, r, \vartheta(\mathbf{z}_n)) \omega_{r+2}(s', [z_{j-2}, z_{j+2}])_p.$$

Moreover,

$$(3.5) \quad \tilde{s}^{(\nu)}(a) = s^{(\nu)}(a) \quad \text{and} \quad \tilde{s}^{(\nu)}(b) = s^{(\nu)}(b), \quad \nu = 0, 1.$$

Proof. This lemma is actually a simpler version of [12, Lemma 1] (see also [16]). It was not proved in [12] for all $p > 0$, and the construction of \tilde{s} was more involved there (because s was allowed to change convexity). For completeness, we recall this construction from [12], adopting it to our case and showing how the estimates can be obtained for all $p > 0$.

With $p_i := s|_{J_i}$, $0 \leq i \leq n - 1$, denote

$$a_i(x) := \frac{(z_{i+2} - z_{i+1})(p'_{i+1}(z_{i+1}) - p'_i(z_{i+1}))}{2(z_{i+1} - z_i)(z_{i+2} - z_i)}(x - z_i)^2, \quad 0 \leq i \leq n - 2,$$

and

$$b_i(x) := \frac{(z_i - z_{i-1})(p'_i(z_i) - p'_{i-1}(z_i))}{2(z_{i+1} - z_i)(z_{i+1} - z_{i-1})}(x - z_{i+1})^2, \quad 1 \leq i \leq n - 1,$$

and also set $a_{n-1}(x) \equiv 0$ and $b_0(x) \equiv 0$. Then, for $x \in [z_i, z_{i+1}]$, $0 \leq i \leq n - 1$, we define

$$\tilde{s}(x) := p_i(x) + a_i(x) + b_i(x).$$

It is now straightforward to verify that s is a continuously differentiable convex function satisfying (3.5), and it remains to prove (3.3) and (3.4). For $1 \leq j \leq n - 1$, we have

$$\begin{aligned} \|s - \tilde{s}\|_{\mathbb{L}_p[z_{j-1}, z_{j+1}]} &\leq 2^{1/p} \max_{i=j-1, j} \|a_i + b_i\|_{\mathbb{L}_p[z_i, z_{i+1}]} \\ &\leq c(p, r, \vartheta(\mathbf{z}_n)) |J_j|^{1+1/p} \max_{i=j-1, j, j+1; 1 \leq i \leq n-1} |p'_i(z_i) - p'_{i-1}(z_i)| \\ &\leq c(p, r, \vartheta(\mathbf{z}_n)) \omega_{r+3}(s, [z_{j-2}, z_{j+2}])_p, \end{aligned}$$

where the last inequality follows from Lemma 3.1. Similarly, Lemma 3.1 implies

$$\begin{aligned} \|s' - \tilde{s}'\|_{\mathbb{L}_p[z_{j-1}, z_{j+1}]} &\leq 2^{1/p} \max_{i=j-1, j} \|a'_i + b'_i\|_{\mathbb{L}_p[z_i, z_{i+1}]} \\ &\leq c(p, r, \vartheta(\mathbf{z}_n)) |J_j|^{1/p} \max_{i=j-1, j, j+1; 1 \leq i \leq n-1} |p'_i(z_i) - p'_{i-1}(z_i)| \\ &\leq c(p, r, \vartheta(\mathbf{z}_n)) \omega_{r+2}(s', [z_{j-2}, z_{j+2}])_p. \quad \square \end{aligned}$$

Lemma 3.8 immediately implies the following result for monotone spline smoothing.

COROLLARY 3.9 (monotone smoothing). *Let $r \in \mathbb{N}_0$, $\mathbf{z}_n := (z_i)_{i=0}^n$, $a =: z_0 < z_1 < \dots < z_{n-1} < z_n := b$ be a partition of $[a, b]$, and let $s \in \Delta^1 \cap \mathcal{S}_{r+1}(\mathbf{z}_n)$. Then, there exists $\tilde{s} \in \Delta^1 \cap \mathcal{S}_{r+1}(\mathbf{z}_n) \cap \mathbb{C}[a, b]$ such that, for any $1 \leq j \leq n-1$ and $0 < p \leq \infty$,*

$$\|s - \tilde{s}\|_{\mathbb{L}_p[z_{j-1}, z_{j+1}]} \leq c(p, r, \vartheta(\mathbf{z}_n))\omega_{r+2}(s, [z_{j-2}, z_{j+2}])_p.$$

Moreover,

$$\tilde{s}(a) = s(a) \quad \text{and} \quad \tilde{s}(b) = s(b).$$

In the case $n = 2$, i.e., when a ppf has only one breakpoint inside an interval $[a, b]$, we get the following corollaries for convex and monotone spline smoothing.

COROLLARY 3.10. *Let $r \in \mathbb{N}_0$, $\mathfrak{z} := (z_i)_{i=0}^2$, $a =: z_0 < z_1 < z_2 := b$ be a partition of $[a, b]$, and let $s \in \Delta^2 \cap \mathcal{S}_{r+2}(\mathfrak{z})$. Then, there exists $\tilde{s} \in \Delta^2 \cap \mathcal{S}_{r+2}(\mathfrak{z}) \cap \mathbb{C}^1[a, b]$ such that*

$$\|s - \tilde{s}\|_{\mathbb{L}_p[a, b]} \leq c(p, r, \vartheta(\mathfrak{z}))\omega_{r+3}(s, [a, b])_p$$

for any $0 < p \leq \infty$. Moreover,

$$\tilde{s}^{(\nu)}(a) = s^{(\nu)}(a) \quad \text{and} \quad \tilde{s}^{(\nu)}(b) = s^{(\nu)}(b), \quad \nu = 0, 1.$$

COROLLARY 3.11. *Let $r \in \mathbb{N}_0$, $\mathfrak{z} := (z_i)_{i=0}^2$, $a =: z_0 < z_1 < z_2 := b$ be a partition of $[a, b]$, and let $s \in \Delta^1 \cap \mathcal{S}_{r+1}(\mathfrak{z})$. Then, there exists $\tilde{s} \in \Delta^1 \cap \mathcal{S}_{r+1}(\mathfrak{z}) \cap \mathbb{C}[a, b]$ such that*

$$\|s - \tilde{s}\|_{\mathbb{L}_p[a, b]} \leq c(p, r, \vartheta(\mathfrak{z}))\omega_{r+2}(s, [a, b])_p$$

for any $0 < p \leq \infty$. Moreover, $\tilde{s}(a) = s(a)$ and $\tilde{s}(b) = s(b)$.

Proof of Theorem 1.2. Let \mathbf{z}_{3n}^* be a refinement of \mathbf{z}_n obtained by adding two extra knots l_i and r_i in each interval $J_i = [z_i, z_{i+1}]$, $0 \leq i \leq n-1$, where $l_i := z_i + |J_i|/4$ and $r_i := z_{i+1} - |J_i|/4$. We now apply Corollaries 3.10 and 3.11 for every $1 \leq i \leq n-1$, with $\mathfrak{z} = (r_{i-1}, z_i, l_i)$ to obtain $s^* \in \mathcal{S}_{q+r}(\mathbf{z}_{3n}^*) \cap \Delta^q \cap \mathbb{C}^{q-1}[-1, 1]$ satisfying

$$\|s - s^*\|_{\mathbb{L}_p[r_{i-1}, l_i]} \leq c(p, r)\omega_{q+r+1}(s, [r_{i-1}, l_i])_p, \quad 1 \leq j \leq n-1.$$

Moreover, $s^* \equiv s$ on $[l_i, r_i]$, $1 \leq i \leq n-2$, and on $[-1, r_0]$ and $[l_{n-1}, 1]$, and therefore

$$(3.6) \quad \|s - s^*\|_{\mathbb{L}_p(J_j)} \leq c(p, r)\omega_{q+r+1}(s, J_j)_p, \quad 0 \leq j \leq n.$$

Now, using Theorem 1.1 with s^* and \mathbf{z}_{3n}^* instead of s and \mathbf{z}_n , respectively, and taking into account that, if $\tilde{\mathbf{z}}_m \in \mathcal{R}_\delta(\mathbf{z}_n)$, then $\tilde{\mathbf{z}}_m \in \mathcal{R}_{4\delta}(\mathbf{z}_{3n}^*)$, we conclude that

$$\|s^* - \tilde{s}\|_{\mathbb{L}_p(\tilde{J}_j)} \leq c(p, q, r)\omega_{q+r+1}(s^*, \tilde{J}_j)_p, \quad 0 \leq j \leq n,$$

where \tilde{J}_j stands for $[(l_{i-1} + r_{i-1})/2, (r_{i-1} + z_i)/2]$, $[(r_{i-1} + z_i)/2, (z_i + l_i)/2]$, or $[(z_i + l_i)/2, (l_i + r_i)/2]$, and $l_i = r_i := -1$ if $i < 0$, and $l_i = r_i := 1$ if $i \geq n$. Finally, we notice that $(l_i + r_i)/2 = (z_i + z_{i+1})/2$ and, hence,

$$\|s^* - \tilde{s}\|_{\mathbb{L}_p(J_j)} \leq c(p, q, r)\omega_{q+r+1}(s^*, J_j)_p, \quad 0 \leq j \leq n,$$

which, together with (3.6), completes the proof of the theorem. \square

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