On asymptotic Lebesgue's universal covering problem

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Lebesgue's universal covering problem

A measurable set $U \subset \mathbb{E}^n$ is called a **universal cover** if it contains a congruent copy of every set $A \subset \mathbb{E}^n$ of diameter 1.

1914: Lebesgue asked Pál in a letter what is the smallest area convex universal cover in the plane ($n = 2$).

Despite many contributions, the problem is still not solved. Records:

2005: Brass and Sharifi established a lower bound of 0.832.

2015: Baez, Bagdasaryan and Gibbs
constructed a convex universal cover of area < 0.8441153.

2018 (pre-print): Gibbs
constructed a convex universal cover of area < 0.8440935944.

Asymptotic Lebesgue's universal covering problem

B_n — the unit ball in $\mathbb{E}^n$
$\mathrm{Vol}(\cdot)$ — the Lebesgue measure in $\mathbb{E}^n$

Jung's theorem: $J_n := r_n B_n$, where $r_n := \sqrt{\frac{n}{2n+2}}$, is a universal cover in $\mathbb{E}^n$.

Our main result: any universal cover has volume at least $(1-o(1))^n \mathrm{Vol}(J_n)$, so J_n **is an asymptotically optimal universal cover**.

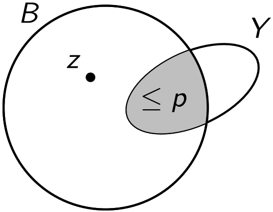
Theorem

Let U be a universal cover in $\mathbb{E}^n$. Then

$$\mathrm{Vol}(U) \geq \exp\left(-\sqrt{\left(\tfrac{5}{4} + o(1)\right) n \log n}\right)\mathrm{Vol}(J_n)\,.$$

Path of the proof: assuming a minimal volume universal cover U is too small, we construct a set of diameter < 1 which cannot be not covered by any congruent copy of U.

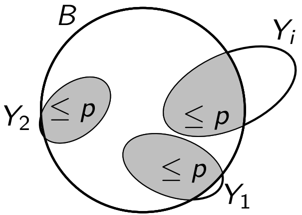
Proof: avoiding a single small set



$\nu(B) = 1$, $\nu(Y \cap B) \leq p$, $0 < p < 1$.

A ν-random single point from B avoids Y with prob. $\geq 1 - p$.

Proof: avoiding sets from a collection



$\nu(B) = 1$, $0 < p < 1$, $\mathcal{Y} = \{Y_i\}$.

$\nu(Y_i \cap B) \leq p$ for each $Y_i \in \mathcal{Y}$.

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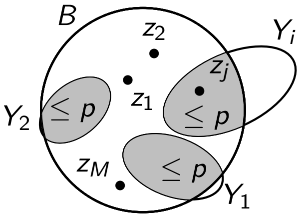

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Take M i.i.d. points in B w.r.t. ν:
$Z := \{z_1, \dots z_M\}$.

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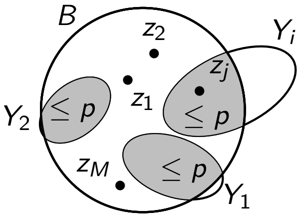

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Then $|Z \cap Y_i| \sim Bi(M, \tilde{p})$, where $\tilde{p} = \nu(Y_i \cap B) \leq p$.

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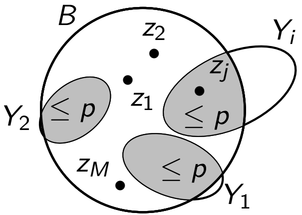

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$\mathbb{P}(|Z \cap Y_i| \geq \frac{M}{2}) < (2ep)^{\frac{M}{2}}$.

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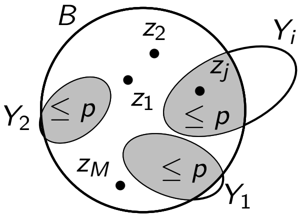

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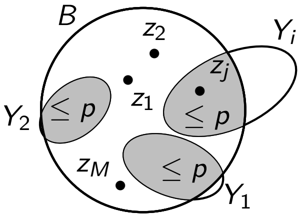

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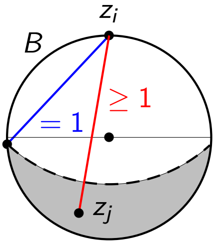
In other words, if $|\mathcal{Y}|(2ep)^{\frac{M}{2}} < \frac{1}{2}$, then with prob. $\geq \frac{1}{2}$
any Y_i covers less than half of Z.

Proof: controlling diameter



B is close to J_n, ν is uniform on B.

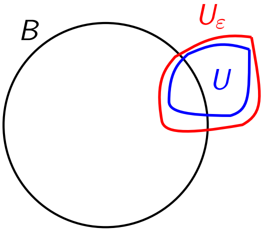
We need $\mathbb{P}(\|z_i - z_j\| \geq 1) < \frac{1}{2M}$.

Among $\binom{M}{2}$ pairs (z_i, z_j) the expected number of "bad" (far) pairs is $< \binom{M}{2}\frac{1}{2M} < \frac{M}{4}$, so by Markov's inequality, with prob. $> \frac{1}{2}$ there is $< \frac{M}{2}$ bad pairs. Removing a point from each pair, we get $X \subset Z$ with $|X| \geq \frac{M}{2}$ and $\mathrm{diam}\,(X) < 1$ while $|X \cap Y_i| < \frac{M}{2}$ for each $Y_i \in \mathcal{Y}$.

Thus X cannot be covered by any $Y_i \in \mathcal{Y}$ and $\mathrm{diam}\,(X) < 1$.

We give a general framework in terms of measurable graphs for constructions cocliques that are hard to cover by members of given family of vertex subsets.

Proof: ε-thickening and discretization of covering family



For a minimal universal cover U, the family $\mathcal{I}_U = \{T(U) : T(U) \cap B \neq \emptyset, T \text{ isometry}\}$ is infinite, so not suitable directly.

With proper $\varepsilon > 0$, we take $U_\varepsilon := U + \varepsilon B_n$ and show $\mathrm{Vol}(U_\varepsilon) < 2\mathrm{Vol}(U)$, and there exists $\mathcal{Y} \subset \mathcal{I}_{U_\varepsilon}$, $|\mathcal{Y}| < \frac{1}{2} n^{n^3}$, such that any set from $\mathcal{I}_U$ is a subset of some $Y \in \mathcal{Y}$. Some ingredients used: a bound on diameter by Makeev, 1990; a bound on size of ε-net in orthogonal group by Szarek, 1981.

Now any subset of B of diameter ≤ 1 is covered by an element of $\mathcal{Y}$.

If $\mathrm{Vol}(U)$ is too small, with suitable choices of the radius of B and ε, p, M, we can probabilstically construct $X \subset B$ as above to obtain a contradiction.

Borsuk's number

Borsuk's number $b(n)$ is the smallest integer such that any set of diameter 1 in $\mathbb{E}^n$ can be covered by $b(n)$ sets of smaller diameter.

$b(n) \geq n+1$ by considering regular simplex in $\mathbb{E}^n$.

1933: Borsuk showed $b(1)=2$, $b(2)=3$, and asked if $b(n)=n+1$, $n \geq 3$?
1947: Perkal $b(3)=4$.

1993: Kahn and Kalai proved $b(n) \geq 1.203^{\sqrt{n}}$ for $n \geq n_0$.
1999: Raigorodskii improved to $b(n) \geq 1.2255^{\sqrt{n}}$ for $n \geq n_0$.

2014: Bondarenko $b(65) > 83$, Jenrich $b(64) > 70$.

1988–89: Schramm; Bourgain and Lindenstrauss: $b(n) \leq \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$.

Universal covers for estimates on Borsuk's number

For small n, partitioning a universal cover into pieces of diameter < 1 can give a good upper bound on $b(n)$.

1920: Pál showed the regular hexagon circumscribed about unit disc is a universal cover, implying $b(2) \leq 3$.

1955: Eggleston; 1957: Grünbaum used suitably truncated octahedra to obtain $b(3) \leq 4$.

1982: Lassak observed $J_n \cap (r_n u + B_n)$, $\|u\| = 1$, is a universal cover, and then proved $b(n) \leq 2^{n-1} + 1$, which remains best known for $4 \leq n \leq 17$.

Corollary

If a universal cover in $\mathbb{E}^n$ is partitioned into M pieces of diameter ≤ 1, then $M \geq (\sqrt{2} - o(1))^n$.

Proof: completion to bodies of constant width, isodiametric inequality and simple volume estimates.

Recall that $b(n) \leq \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$ Schramm; Bourgain and Lindenstrauss.

Asymptotic optimality of J_n for other measures

Mean width $w(K)$ of a convex body K in $\mathbb{E}^n$ is half the average length of a projection of K on a line, e.g. $w(B_n) = 1$.

1990 Makeev; 1998 Bezdek and Connelly:
J_n minimizes **mean width** of **translative** universal covers.

For the general congruent (non-translative) universal covers J_n is not a minimizer of mean width or volume: $J_n \cap (r_n u + B_n)$, $\|u\| = 1$, is smaller.

Corollary

$w(U) \geq (1 - o(1))w(J_n)$ for any convex universal cover U.

Proof: Urysohn's inequality $w(U) \geq \left(\frac{\mathrm{Vol}(U)}{\mathrm{Vol}(B_n)}\right)^{\frac{1}{n}}$.

Remark

Using Alexandrov inequality, the corollary can be extended to all quermassintegrals/intrinsic volumes, e.g. to the surface area.

More details at `https://arxiv.org/abs/2512.04023`.

Thank you!