Polynomial approximation on domains with C^2 boundary

Andriy Prymak

joint work with Feng Dai

Let Ω be the closure of an open, connected and bounded domain with C^2 boundary in $\mathbb{R}^d$.
We introduce a computable modulus of smoothness $\omega_\Omega^r(f,t)_p$ satisfying the following two results.

Theorem (direct). *If $r, n \in \mathbb{N}$, $0 < p \leq \infty$ and $f \in L^p(\Omega)$, then*

$$E_n(f)_{L^p(\Omega)} \leq C\omega_\Omega^r(f, n^{-1})_p,$$

where the constant C is independent of f and n.

Theorem (inverse). *If $r, n \in \mathbb{N}$, $1 \leq p \leq \infty$ and $f \in L^p(\Omega)$, then*

$$\omega_\Omega^r(f, n^{-1})_p \leq \frac{C}{n^r} \sum_{j=0}^{n} (j+1)^{r-1} E_j(f)_{L^p(\Omega)},$$

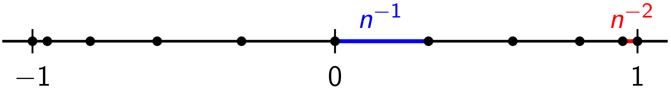
where the constant C is independent of f and n.

For the one-dimensional case $\Omega = [-1, 1]$, this is a classical result by Ditzian and Totik. The modulus is defined as

$$\omega^r_{[-1,1]}(f, n^{-1})_p = \sup_{0<t<n^{-1}} \left\| \widetilde{\Delta}^r_{t\varphi(\cdot)} f(\cdot) \right\|_{L^p([-1,1])}, \quad \varphi(x) := \sqrt{1-x^2}.$$

If $x_j = \cos(j\pi/n)$, $0 \le j \le n$, the above modulus is equivalent to the following local modulus:

$$\omega^r_{[-1,1],loc}(f, n^{-1})_p = \left(\sum_{j=1}^{n-1} \sup_{0<rt<x_{j-1}-x_{j+1}} \| \Delta^r_t f \|^p_{L^p([x_{j+1}, x_{j-1}-rt])} \right)^{1/p}.$$



Ivanov, 1984: announcement for piecewise C^2 planar domains, partial results for inverse.

Ditzian, Totik, 1987: simple polytopes for $1 \leq p \leq \infty$.

Netrusov, 1994: announcement of characterization of approximation classes.

Dubiner, 1995: possibly reduction to local approximation.

Ditzian, 1996: simple polytopes for $0 < p \leq \infty$.

Dai, Xu, 2010: ball (and sphere).

Ditzian, 2014: ball (and sphere) – a different approach.

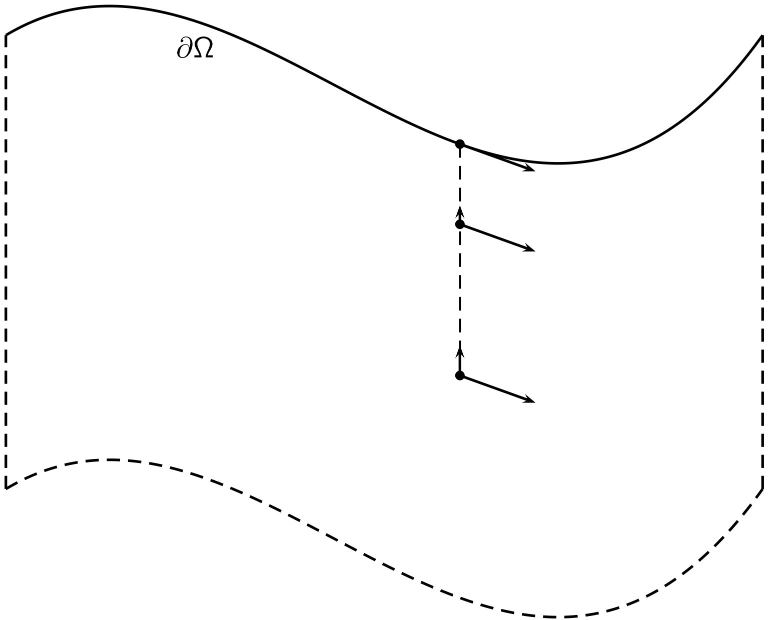
Totik, 2014: general polytopes in $\mathbb{R}^3$ for $1 \leq p < \infty$,
general polytopes for $p = \infty$ (using an incorrect claim for $d \geq 4$).

Totik, $\geq$2019: general polytopes and algebraic domains for $p = \infty$.

Our computable modulus of smoothness $\omega^r_\Omega(f,t)_p$ is the sum of two moduli.

1. **Directional** modulus, which is the direct generalization of Ditzian-Totik modulus along the directions of the coordinate axes. The step of the finite difference is proportional to the root of the distance to the boundary along the considered direction.

2. **Tangential** modulus, which is defined on domains of special type that are appropriately truncated subgraphs of functions defining $\partial\Omega$. The size of the step of the finite difference is uniform, but the direction is parallel to the corresponding tangential directions of $\partial\Omega$.


$\partial\Omega$

In a somewhat simplified form, for a domain $G \subset \Omega \subset \mathbb{R}^2$ of special type

$$G = \{(x, y) : -1 \leq x \leq 1,\ g(x) - 1 \leq y \leq g(x)\},$$

where $g \in C^2$, the corresponding term of the tangential modulus is

$$\widetilde{\omega}_G^r(f, t)_p :=$$
$$\sup_{0<s\leq t} \Big(\int_{-1}^{1} \int_{-1}^{-A_0 t^2} \Big[\frac{1}{t} \int_{x-t}^{x+t} |\Delta_{s\xi(u)}^r (f, \Omega, (x, g(x) + z))|^p \, du \Big] \, dz \, dx \Big)^{\frac{1}{p}},$$

where the tangential direction is given by $\xi(u) := (1, g'(u))$.

For $r, n \in \mathbb{N}$, $0 < p \le \infty$ and $f \in L^p(\Omega)$, we prove the **direct result**

$$E_n(f)_{L^p(\Omega)} \le C\omega^r_{\Omega,loc}(f, n^{-1})_p,$$

where $\omega^r_{\Omega,loc}$ is certain local modulus satisfying

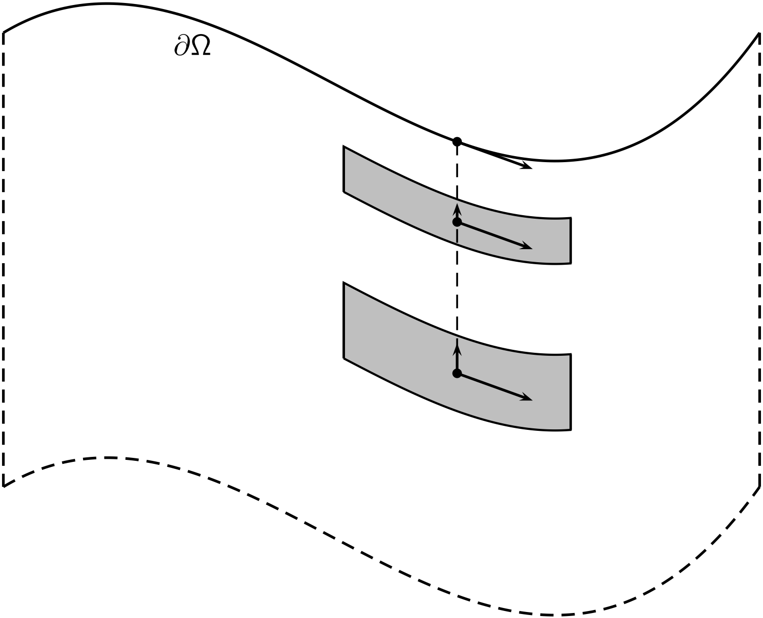
$$\omega^r_{\Omega,loc}(f, n^{-1})_p \le C\omega^r_{\Omega}(f, n^{-1})_p.$$

The definition of the local modulus uses p-mean of the directional moduli of smoothness over $\approx n^d$ small subdomains.

In each subdomain we only need one coordinate direction and $d-1$ tangential directions.

Therefore, $\omega^r_{\Omega,loc}(f, n^{-1})_p$ depends only on a finite (depending on n) number of tangential directions.

However, the points where the tangential directions are taken need to be selected in a certain way to ensure $\omega^r_{\Omega,loc}(f, n^{-1})_p \le C\omega^r_{\Omega}(f, n^{-1})_p$.


∂Ω

Direct result: we use local approximations derived from a **Whitney-type** result and patch them together using appropriate **partition of unity**.

Namely, we decompose Ω as the union of smaller pieces I_j and construct the approximating polynomial in the form

$$P(x) = \sum_j p_j(x) q_j(x),$$

where p_j is a polynomial of small degree approximating f well on I_j and q_j is a polynomial approximating the indicator function of I_j well, $\sum_j q_j = 1$.

The proper form of the multivariate **Whitney inequality** using directional modulus of smoothness is

$$\inf_P \|f - P\|_{L^p(G)} \le c\omega_G^r(f, \mathrm{diam}(G), \mathcal{E})_p,$$

where the infimum is taken over all polynomials P s.t. $P(x + te)$ is a polynomial of degree $< r$ in $t \in \mathbb{R}$ for any fixed $x \in \mathbb{R}^d$ and $e \in \mathcal{E} \subset \mathbb{S}^{d-1}$.

We offer a general procedure how one can use Whitney inequality on one domain and obtain Whitney inequality on a bigger domain of certain structure.

This allows us to get the needed local polynomial approximations on not necessarily convex pieces and use the moduli which depend on few directions.

We also obtained a different proof of the result by Dekel and Leviatan for Whitney inequalities on convex domains.

The construction of the **partition of unity** strongly relies on the ideas of Dzjadyk and Konovalov, 1973.

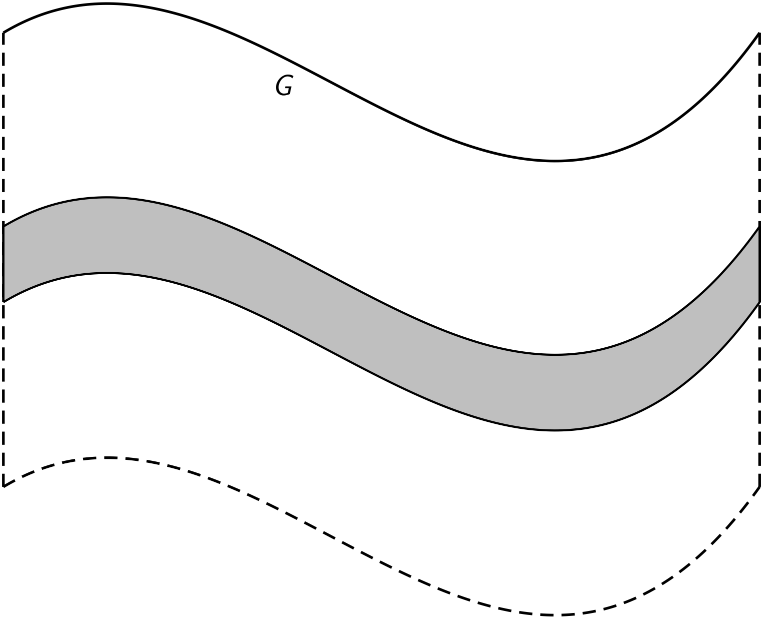
To obtain an appropriate partition of unity on domains of special type

$$G = \{(x, y) : -1 \leq x \leq 1,\ g(x) - 1 \leq y \leq g(x)\},$$

where $g \in C^2$, the key step is to approximate the characteristic function of

$$\{(x, y) : -1 \leq x \leq 1,\ g(x) - a_i \leq y \leq g(x) - a_{i+1}\}$$

with $\{a_i\}$ being the Chebyshev nodes on $[0, 2]$.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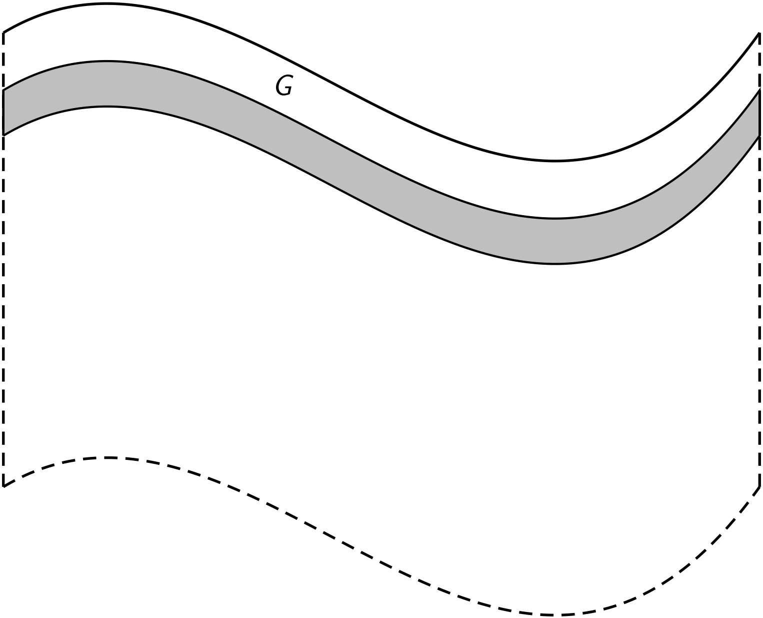
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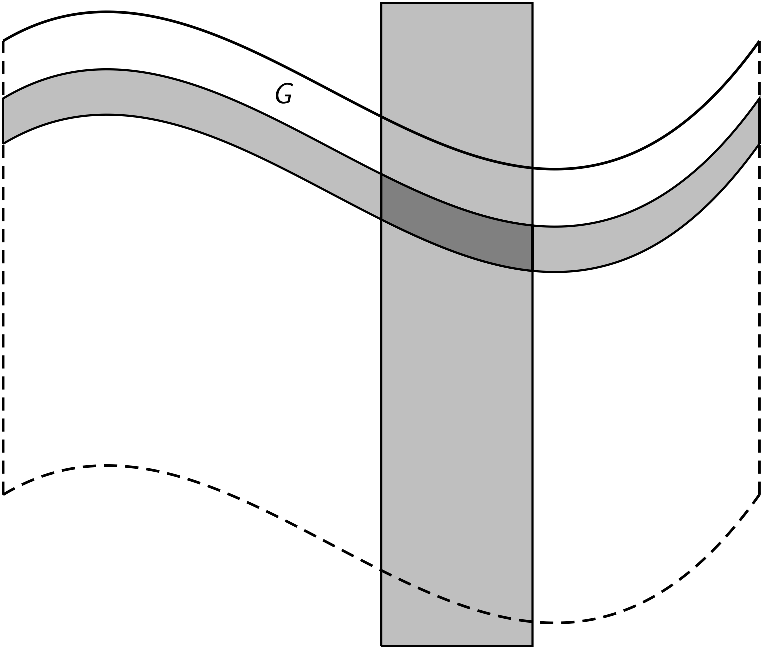

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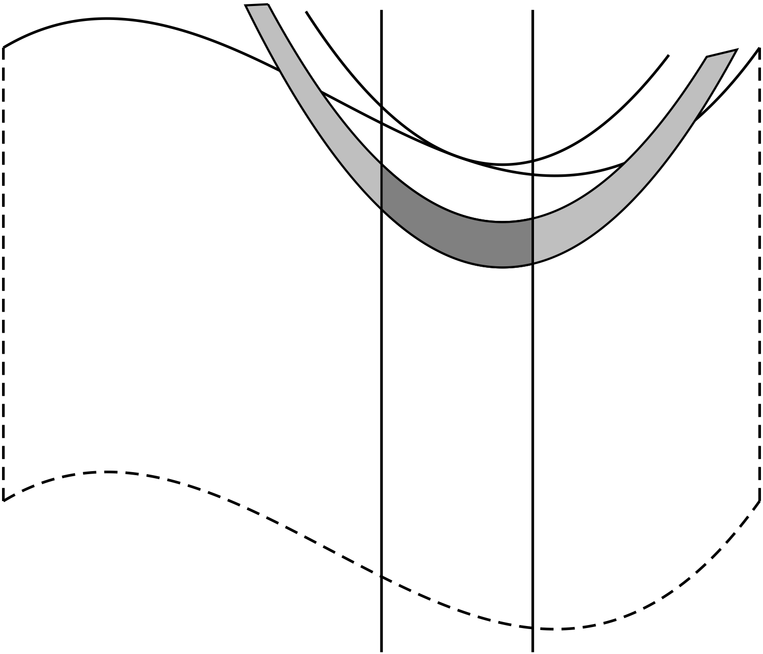

Inverse result: follows from the new tangential Bernstein inequality.

For domains $G_1 \subset G_2 \subset \mathbb{R}^2$ of special type

$$G_t = \{(x, y) : -t \le x \le t,\ g(x) - t \le y \le g(x)\},$$

where $g \in C^2$, with $\varphi_n(x, y) := n^{-1} + \sqrt{g(x) - y}$,

$$\mathcal{D}^\ell_{x_0} := (\partial_1 + g'(x_0)\partial_2)^\ell \quad \text{and} \quad \mathcal{D}^\ell f(x, y) := (\mathcal{D}^\ell_x f)(x, y),$$

we proved for any $0 < p \le \infty$, $i, j, r = 0, 1, 2, \ldots$ and any polynomial P_n of total degree $\le n$ that

$$\|\varphi_n^i \mathcal{D}^r \partial_2^{i+j} P_n\|_{L^p(G_1)} \le c n^{r+i+2j} \|P_n\|_{L^p(G_2)}.$$

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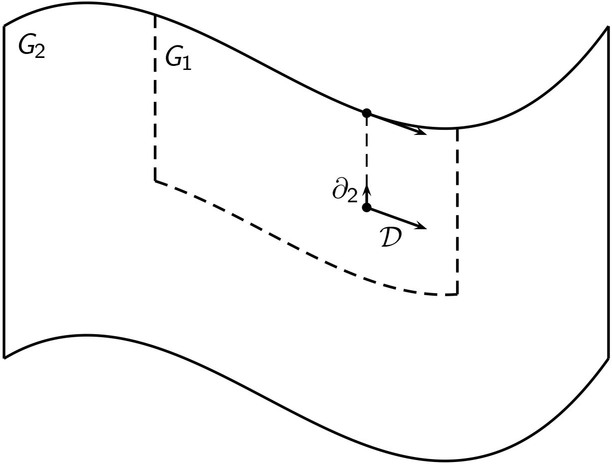
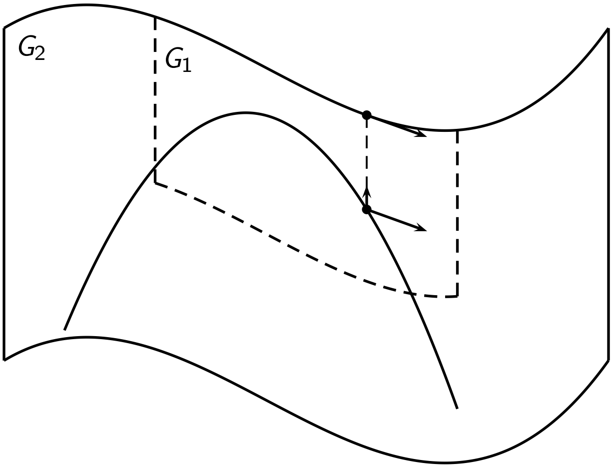


$$\|\varphi_n^i \mathcal{D}^r \partial_2^{i+j} P_n\|_{L^p(G_1)} \leq c n^{r+i+2j} \|P_n\|_{L^p(G_2)}$$



The main idea is a non-trivial reduction to the use of weighted one-dimensional Bernstein inequalities along suitable families of parabolas contained in Ω.

Consider the following metric on Ω:

$$\rho_\Omega(\xi, \mu) := \|\xi - \mu\| + \left|\sqrt{\operatorname{dist}(\xi, \partial\Omega)} - \sqrt{\operatorname{dist}(\mu, \partial\Omega)}\right|, \quad \xi, \mu \in \Omega,$$

and for $t > 0$ set $U(\xi, t) := \{\mu \in \Omega : \rho_\Omega(\xi, \mu) \le t\}$. Following Ivanov, for $0 < q \le p \le \infty$ and $f \in L^p(\Omega)$, the averaged modulus is defined as

$$\tau_r(f; \delta)_{p,q} := \left\| w_r(f, \cdot, \delta)_q \right\|_p,$$

where

$$w_r(f, \xi, \delta)_q := \left(\frac{1}{|U(\xi, \delta)|} \int_{U(\xi,\delta)} |\Delta^r_{(\eta-\xi)/r}(f, \Omega, \xi)|^q \, d\eta \right)^{\frac{1}{q}}.$$

We show that for any $0 < q \le p \le \infty$ our modulus satisfies

$$\omega^r(f, n^{-1})_p \le C\tau_r(f, An^{-1})_{p,q}.$$

Next, if $r, n \in \mathbb{N}$, $1 \le p \le \infty$ and $f \in L^p(\Omega)$, we obtain the inverse inequality

$$\tau_r(f, n^{-1})_{p,q} \le \frac{C}{n^r} \sum_{j=0}^{n} (j+1)^{r-1} E_j(f)_{L^p(\Omega)}.$$

In summary, we:

- considered three different moduli: local, computable, averaged;
- established ascending order;
- proved direct estimate for the smallest (local) modulus;
- proved inverse estimate for the largest (averaged) modulus.

This gives matching direct and inverse inequalities for any of the three moduli.

Other applications of the new tangential Bernstein inequality include a Marcinkiewicz-type discretization result for $p \geq 1$

$$\left(\sum_{\xi \in \Lambda} \lambda_\xi |P_n(\xi)|^p\right)^{1/p} \approx \|P_n\|_{L_p(\Omega)},$$

and a positive cubature formula

$$\sum_{\xi \in \Lambda} \lambda_\xi P_n(\xi) = \int_\Omega P_n(\mu) d\mu,$$

both valid for any polynomial P_n of total degree $\leq n$, where Λ has cardinality of order n^d and is a maximal $\frac{c}{n}$-separated set w.r.t. the metric ρ_Ω, while λ_ξ is up to a constant the volume of the $\frac{c}{n}$-negihborhood of ξ w.r.t. ρ_Ω.

Open problems:

- less smooth domains;
- inverse inequality for $p < 1$;
- K-functional;
- equivalence of various moduli;
- questions related to directional Whitney inequalities;
- approximation with weights;
- convexity preserving approximation.

Thank you!

Email prymak@gmail.com to be notified when the pre-print is available.