Polynomial approximation on domains with C^2 boundary

Andriy Prymak

joint work with Feng Dai

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$$E_n(f)_{L^p(\Omega)} \leq C\omega^r_\Omega(f, n^{-1})_p,$$

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Theorem (inverse). *If $r, n \in \mathbb{N}$, $1 \le p \le \infty$ and $f \in L^p(\Omega)$, then*

$$\omega_\Omega^r(f, n^{-1})_p \le \frac{C}{n^r} \sum_{j=0}^{n} (j+1)^{r-1} E_j(f)_{L^p(\Omega)},$$

where the constant C is independent of f and n.

For the one-dimensional case $\Omega = [-1,1]$, this is a classical result by Ditzian and Totik. The modulus is defined as

$$\omega^r_{[-1,1]}(f, n^{-1})_p = \sup_{0<t<n^{-1}} \left\| \widetilde{\Delta}^r_{t\varphi(\cdot)} f(\cdot) \right\|_{L^p([-1,1])}, \quad \varphi(x) := \sqrt{1-x^2}.$$

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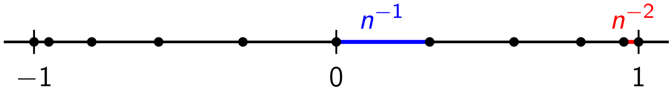
If $x_j = \cos(j\pi/n)$, $0 \le j \le n$, the above modulus is equivalent to the following local modulus:

$$\omega^r_{[-1,1],loc}(f, n^{-1})_p = \left(\sum_{j=1}^{n-1} \sup_{0<rt<x_{j-1}-x_{j+1}} \|\Delta^r_t f\|^p_{L^p([x_{j+1}, x_{j-1}-rt])} \right)^{1/p}.$$



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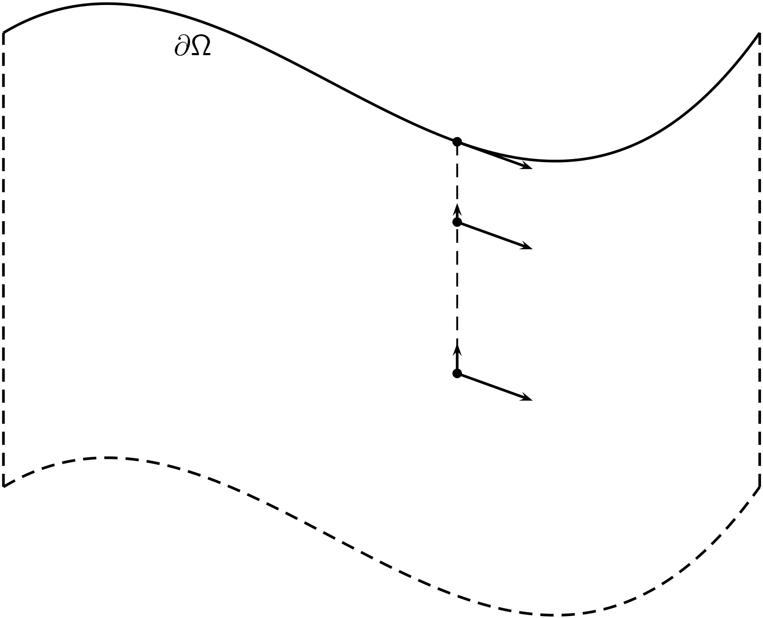
1. **Directional** modulus, which is the direct generalization of Ditzian-Totik modulus along the directions of the coordinate axes. The step of the finite difference is proportional to the root of the distance to the boundary along the considered direction.

2. **Tangential** modulus, which is defined on domains of special type that are appropriately truncated subgraphs of functions defining $\partial\Omega$. The size of the step of the finite difference is uniform, but the direction is parallel to the corresponding tangential directions of $\partial\Omega$.


$\partial\Omega$

In a somewhat simplified form, for a domain $G \subset \Omega \subset \mathbb{R}^2$ of special type

$$G = \{(x, y) : -1 \le x \le 1,\ g(x) - 1 \le y \le g(x)\},$$

where $g \in C^2$, the corresponding term of the tangential modulus is

$$\widetilde{\omega}^r_G(f, t)_p := \sup_{0<s\le t} \Big(\int_{-1}^{1} \int_{-1}^{-A_0 t^2} \Big[\frac{1}{t} \int_{x-t}^{x+t} |\Delta^r_{s\xi(u)}(f, \Omega, (x, g(x) + z))|^p \, du \Big] \, dz \, dx \Big)^{\frac{1}{p}},$$

where the tangential direction is given by $\xi(u) := (1, g'(u))$.

For $r, n \in \mathbb{N}$, $0 < p \leq \infty$ and $f \in L^p(\Omega)$, we prove the **direct result**

$$E_n(f)_{L^p(\Omega)} \leq C\omega^r_{\Omega,loc}(f, n^{-1})_p,$$

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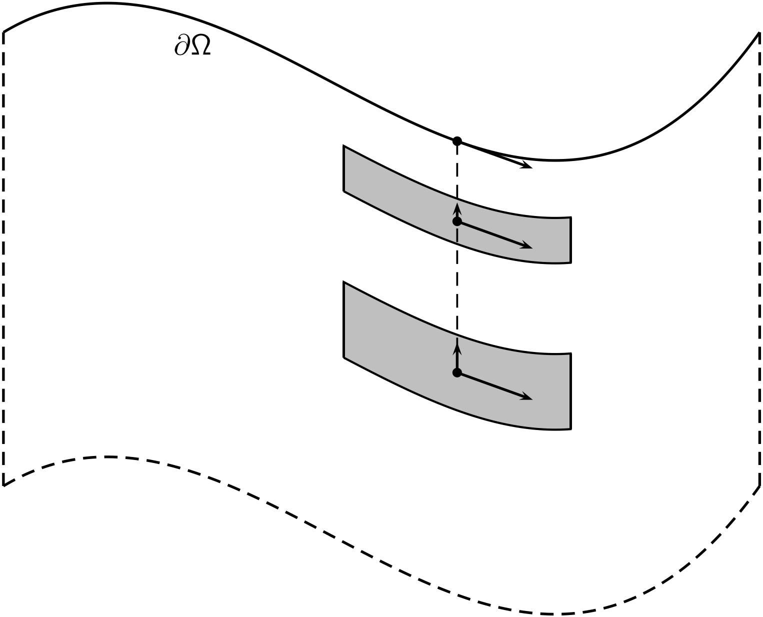
Therefore, $\omega^r_{\Omega,loc}(f, n^{-1})_p$ depends only on a finite (depending on n) number of tangential directions.


∂Ω

Direct result: we use local approximations derived from a **Whitney-type** result and patch them together using appropriate **partition of unity**.

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Namely, we decompose Ω as the union of smaller pieces I_j and construct the approximating polynomial in the form

$$P(x) = \sum_j p_j(x) q_j(x),$$

where p_j is a polynomial of small degree approximating f well on I_j and q_j is a polynomial approximating the indicator function of I_j well, $\sum_j q_j = 1$.

The proper form of the multivariate **Whitney inequality** using directional modulus of smoothness is

$$\inf_P \|f - P\|_{L^p(G)} \leq c\omega_G^r(f, \mathrm{diam}(G), \mathcal{E})_p,$$

where the infimum is taken over all polynomials P s.t. $P(x+te)$ is a polynomial of degree $< r$ in $t \in \mathbb{R}$ for any fixed $x \in \mathbb{R}^d$ and $e \in \mathcal{E} \subset \mathbb{S}^{d-1}$.

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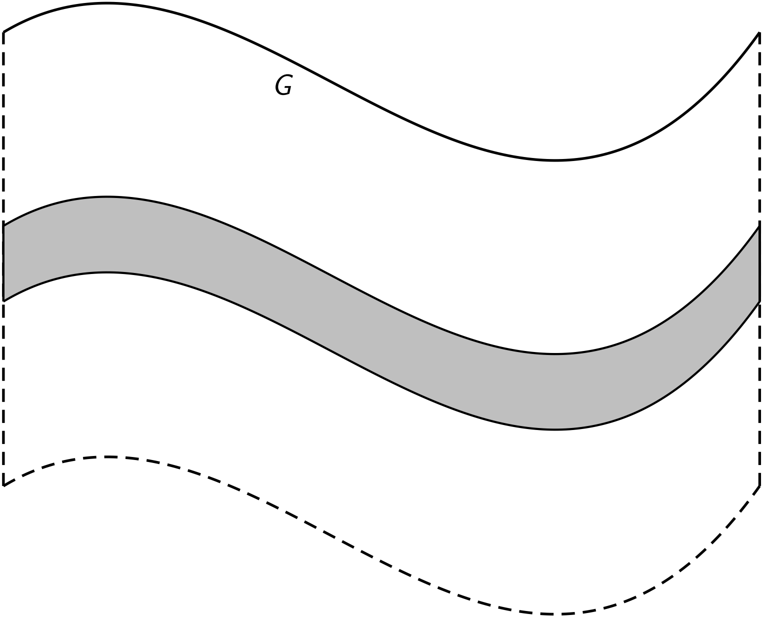
To obtain an appropriate partition of unity on domains of special type

$$G = \{(x, y) : -1 \leq x \leq 1,\ g(x) - 1 \leq y \leq g(x)\},$$

where $g \in C^2$, the key step is to approximate the characteristic function of

$$\{(x, y) : -1 \leq x \leq 1,\ g(x) - a_i \leq y \leq g(x) - a_{i+1}\}$$

with $\{a_i\}$ being the Chebyshev nodes on $[0, 2]$.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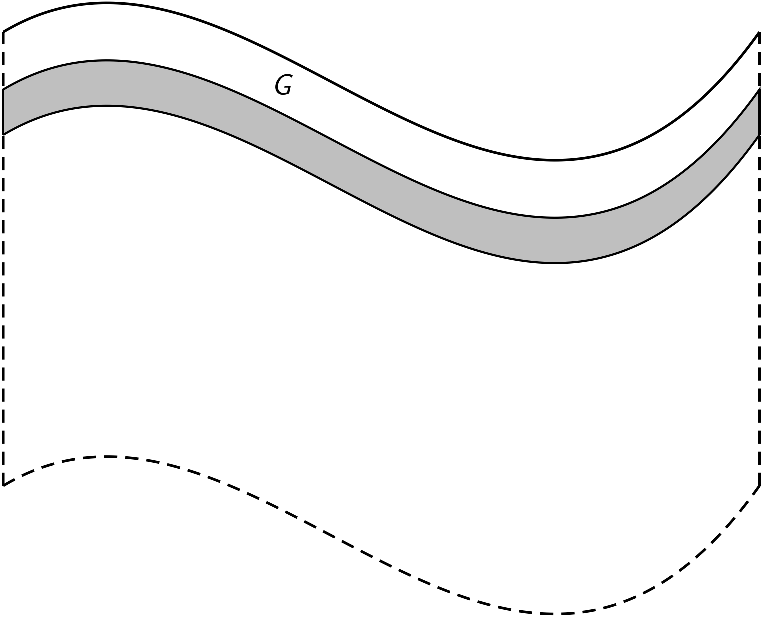
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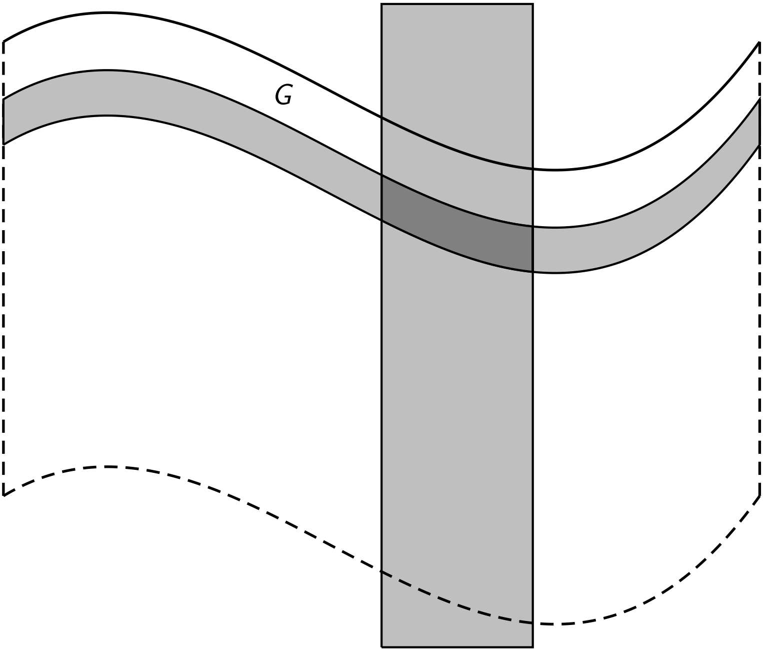
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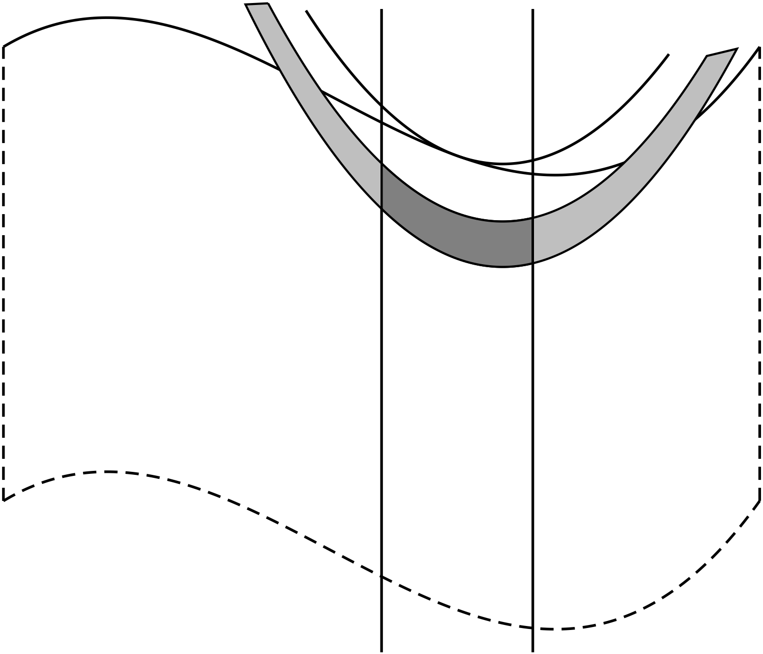

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For domains $G_1 \subset G_2 \subset \mathbb{R}^2$ of special type

$$G_t = \{(x, y) : -t \le x \le t,\ g(x) - t \le y \le g(x)\},$$

where $g \in C^2$, with $\varphi_n(x, y) := n^{-1} + \sqrt{g(x) - y}$,

$$\mathcal{D}^{\ell}_{x_0} := (\partial_1 + g'(x_0)\partial_2)^{\ell} \quad \text{and} \quad \mathcal{D}^{\ell} f(x, y) := (\mathcal{D}^{\ell}_{x} f)(x, y),$$

we proved for any $0 < p \le \infty$, $i, j, r = 0, 1, 2, \ldots$ and any polynomial P_n of total degree $\le n$ that

$$\|\varphi_n^i \mathcal{D}^r \partial_2^{i+j} P_n\|_{L^p(G_1)} \le c n^{r+i+2j} \|P_n\|_{L^p(G_2)}.$$

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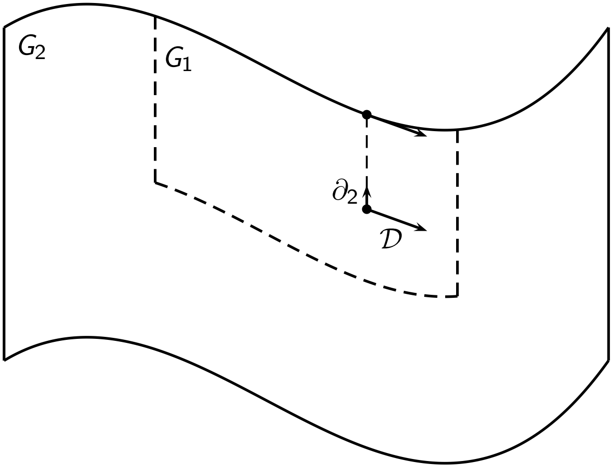

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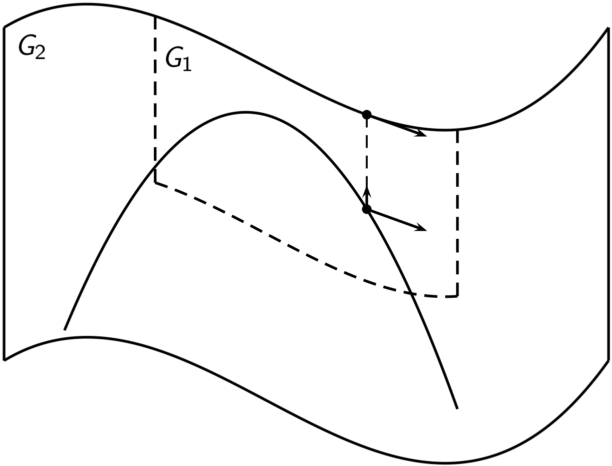

The main idea is a non-trivial reduction to the use of weighted one-dimensional Bernstein inequalities along suitable families of parabolas contained in Ω.

Consider the following metric on Ω:

$$\rho_\Omega(\xi,\mu) := \|\xi-\mu\| + \left|\sqrt{\mathrm{dist}(\xi,\partial\Omega)} - \sqrt{\mathrm{dist}(\mu,\partial\Omega)}\right|, \quad \xi,\mu\in\Omega,$$

and for $t>0$ set $U(\xi,t) := \{\mu\in\Omega : \rho_\Omega(\xi,\mu)\le t\}$. Following Ivanov, for $0<q\le p\le\infty$ and $f\in L^p(\Omega)$, the averaged modulus is defined as

$$\tau_r(f;\delta)_{p,q} := \left\| w_r(f,\cdot,\delta)_q \right\|_p,$$

where

$$w_r(f,\xi,\delta)_q := \Big(\frac{1}{|U(\xi,\delta)|}\int_{U(\xi,\delta)} |\Delta^r_{(\eta-\xi)/r}(f,\Omega,\xi)|^q \, d\eta\Big)^{\frac{1}{q}}.$$

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We show that for any $0 < q \le p \le \infty$ our modulus satisfies

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Next, if $r,n\in\mathbb{N}$, $1\le p\le\infty$ and $f\in L^p(\Omega)$, we obtain the inverse inequality

$$\tau_r(f,n^{-1})_{p,q} \le \frac{C}{n^r}\sum_{j=0}^{n}(j+1)^{r-1}E_j(f)_{L^p(\Omega)}.$$

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This gives matching direct and inverse inequalities for any of the three moduli.

Other applications of the new tangential Bernstein inequality include a Marcinkiewicz-type discretization result for $p \geq 1$

$$\left(\sum_{\xi \in \Lambda} \lambda_\xi |P_n(\xi)|^p \right)^{1/p} \approx \|P_n\|_{L_p(\Omega)},$$

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$$\left(\sum_{\xi \in \Lambda} \lambda_\xi |P_n(\xi)|^p \right)^{1/p} \approx \|P_n\|_{L_p(\Omega)},$$

and a positive cubature formula

$$\sum_{\xi \in \Lambda} \lambda_\xi P_n(\xi) = \int_\Omega P_n(\mu) d\mu,$$

both valid for any polynomial P_n of total degree $\leq n$, where Λ has cardinality of order n^d and is a maximal $\frac{c}{n}$-separated set w.r.t. the metric ρ_Ω, while λ_ξ is up to a constant the volume of the $\frac{c}{n}$-negihborhood of ξ w.r.t. ρ_Ω.

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- convexity preserving approximation.

Thank you!

Thank you!

Email prymak@gmail.com to be notified when the pre-print is available.