Convex bodies of constant width with exponential illumination number

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Borsuk's number

Borsuk's number $b(n)$ is the smallest integer such that any set of diameter 1 in $\mathbb{E}^n$ can be covered by $b(n)$ sets of smaller diameter.

$b(n) \geq n+1$ by considering regular simplex in $\mathbb{E}^n$.

Borsuk (1933) asked if $b(n) = n+1$ for all n?
Borsuk (1933): $b(1) = 2$ and $b(2) = 3$,
Perkal (1947): $b(3) = 4$.

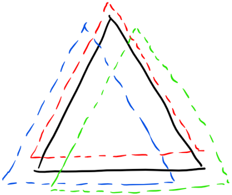
Asymptotic lower bound: $b(n) \geq c^{\sqrt{n}}$ for large n established by
Kahn and Kalai (1993): $c \approx 1.203$,
Raigorodskii (1999): $c \approx 1.2255$.

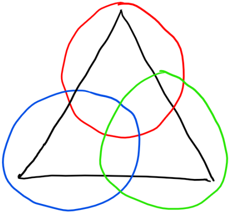
Smallest known n with $b(n) > n+1$ is $n = 64$.

Asymptotic upper bound on $b(n)$

Schramm (1988), Bourgain and Lindenstrauss (1989):

$$b(n) \leq \left(\sqrt{\frac{3}{2}} + o(1) \right)^n$$





Bourgain and Lindenstrauss's results

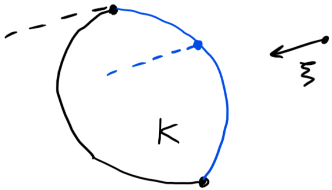
Let $g(n)$ be the smallest number of balls of diameter < 1 needed to cover an arbitrary set of diameter 1 in $\mathbb{E}^n$. Clearly, $b(n) \le g(n)$.

Rogers (1965): $g(n) \le (\sqrt{2} + o(1))^n$
Danzer (1965): $g(n) \ge 1.003^n$

Bourgain and Lindenstrauss (1989): $1.0645^n \le g(n) \le \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$.

Illumination and covering

Let K be a convex body in $\mathbb{E}^n$. A point $x \in \partial K$ is illuminated by a direction $\xi \in \mathbb{S}^{n-1}$ if the ray $\{x + \xi t : t \geq 0\}$ intersects $\text{int}(K)$.



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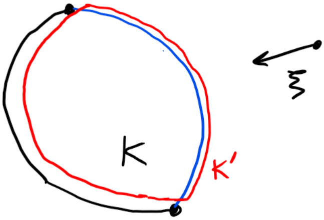
The illumination number $I(K)$ is the minimal number of directions such that every $x \in \partial K$ is illuminated by one of these directions.

Denote $h(K)$ to be the smallest number N such that K can be covered by N smaller homothetic copies of K.
Boltyanski (1960): $I(K) = h(K)$ for any convex body K.



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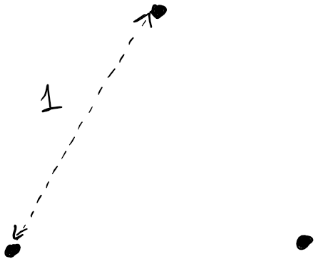
Levi-Hadwiger-Gohberg-Markus's conjecture: $I(K) = h(K) \leq 2^n$
with equality iff K is an affine copy of a cube.

Convex bodies of constant width

A convex body in $\mathbb{E}^n$ has constant width, if its projection onto any line has the same length. It is well-known that any set of diameter 1 is contained in a convex body of constant width 1.



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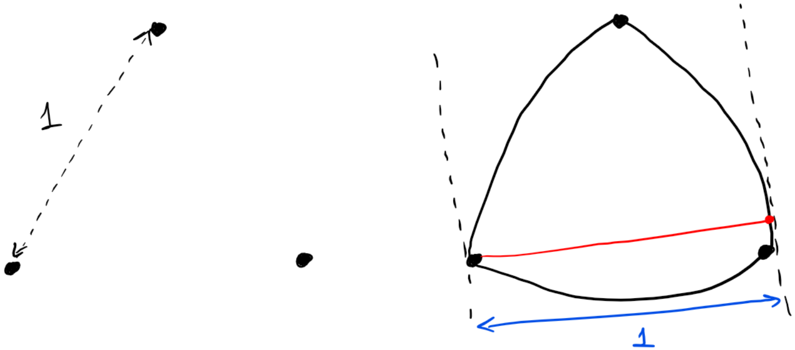


Therefore, it suffices to consider only bodies of constant width when computing the Borsuk's number $b(n)$.

Schramm's upper bound on Borsuk's number

Define

$$h(n) := \sup\{h(K) = I(K) : K \text{ is a convex body of constant width in } \mathbb{E}^n\}.$$

We have $b(n) \leq h(n)$.

Schramm (1988): $h(n) \leq \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$

The only known lower bound on $h(n)$ was the same as for $b(n)$:
$h(n) \geq b(n) \geq 1.2255^{\sqrt{n}}$ for large n.

Kalai (2015) asked: does there exist $C > 1$ with $h(n) \geq C^n$ for large n?

Main result

We answer the question of Kalai in the affirmative.

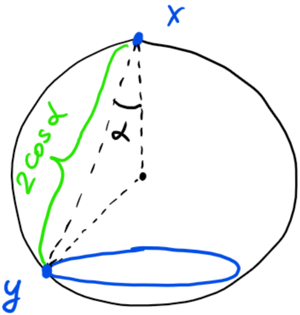
Theorem 1

$$h(n) \geq \frac{c}{\sqrt{n}\log n}\left(\frac{1}{\cos(\pi/14)}\right)^{n}$$

Main geometric ingredient

For fixed $x \in \mathbb{S}^{n-1}$ and $0 < \alpha \leq \pi/6$ define

$$Q(x, \alpha) := \{x\} \cup \{y \in \mathbb{S}^{n-1} : \|x - y\| = 2\cos\alpha\}.$$



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For non-zero $x, y \in \mathbb{E}^n$, let

$$\theta(x, y) := \arccos(\tfrac{x \cdot y}{\|x\|\|y\|}).$$

For $x \in \mathbb{S}^{n-1}$ and $0 < \alpha < \pi$, set

$$C(x, \alpha) := \{y \in \mathbb{S}^{n-1} : \theta(x, y) \leq \alpha\}.$$

Lemma 1

Suppose $0 < \alpha \leq \pi/6$, K is a convex body in $\mathbb{E}^n$ s.t. $\operatorname{diam} K = 2\cos\alpha$ and for some $x \in \mathbb{S}^{n-1}$ we have $Q(x, \alpha) \subset K$. Then $x \in \partial K$ and any direction $\xi \in \mathbb{S}^{d-1}$ illuminating x satisfies $\xi \in C(-x, \frac{\pi}{2} - \alpha)$.

Main geometric ingredient

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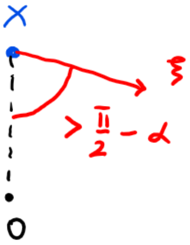
Suppose $0 < \alpha \leq \pi/6$*,* K *is a convex body in* $\mathbb{E}^n$ *s.t.* $\operatorname{diam} K = 2\cos\alpha$ *and for some* $x \in \mathbb{S}^{n-1}$ *we have* $Q(x, \alpha) \subset K$*. Then* $x \in \partial K$ *and any direction* $\xi \in \mathbb{S}^{d-1}$ *illuminating* x *satisfies* $\xi \in C(-x, \frac{\pi}{2} - \alpha)$*.*



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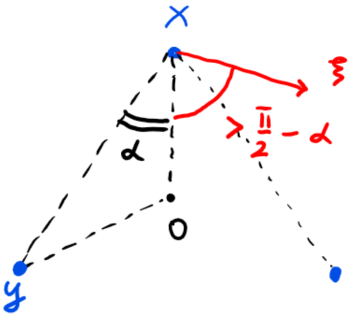

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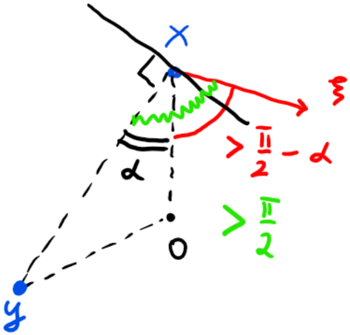

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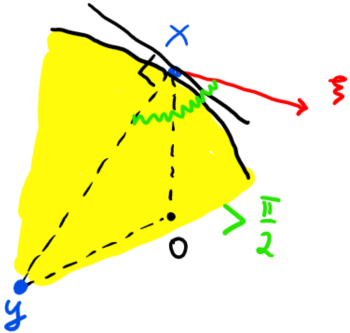

Separation required to control the diameter

For a finite $X \subset \mathbb{S}^{n-1}$, let $\mathcal{W}(X) := \bigcup_{x \in X} Q(x, \alpha)$.

Lemma 2

Suppose $0 < \alpha \leq \pi/6$ *and* $X \subset \mathbb{S}^{n-1}$.

(i) *If* $\theta(x, y) \leq \pi - 2\alpha$ *for all* $x, y \in X$, *then* $\operatorname{diam} X \leq 2\cos\alpha$.

(ii) *If* $4\alpha \leq \theta(x, y) \leq \pi - 6\alpha$ *for all distinct* $x, y \in X$, *then* $\operatorname{diam} \mathcal{W}(X) \leq 2\cos\alpha$.

Thinly spread subsets of the sphere

Lemma 3

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists a collection* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|\{i : x \in C(x_i, \varphi)\}| \leq Cn \log n$ *for all* $x \in \mathbb{S}^{n-1}$.

If μ denotes the spherical probability measure on $\mathbb{S}^{n-1}$, then up to a constant factor $\mu(C(x_i, \varphi))$ behaves like $\frac{(\sin\varphi)^n}{\sqrt{n}}$ for large n.

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(b) $|\{i : x \in C(x_i, \varphi)\}| \leq Cn \log n$ *for all* $x \in \mathbb{S}^{n-1}$*.*

Proof outline: Sample an appropriately selected number of uniformly i.i.d. points from $\mathbb{S}^{n-1}$. By Böröczky and Wintsche (2003), which is the adaptation of the ideas of Erdős and Rogers (1961/62) to $\mathbb{S}^{n-1}$, the resulting set Y satisfies (b) with high probability.
Certain probabilistic arguments show that some points that may violate (a) can be removed from Y to obtain the desired $X \subset Y$.

Proof of the main result

Theorem 1

$$h(n) \geq \frac{c}{\sqrt{n}\log n}\left(\frac{1}{\cos(\pi/14)}\right)^n$$

Proof: Use Lemma 3 with $\varphi = \frac{6\pi}{14}$ to get a thinly spread $X \subset \mathbb{S}^{n-1}$.
Construct $\mathcal{W}(X) = \bigcup_{x \in X} Q(x, \alpha)$ with $\alpha = \frac{\pi}{14}$.
By Lemma 2 (ii) (separation lemma), $\operatorname{diam}(\mathcal{W}(X)) = 2\cos\alpha$.
So there exists a body $K \supset \mathcal{W}(X)$ of constant width $2\cos\alpha$.
Since $\varphi = \frac{\pi}{2} - \alpha$, Lemma 3 (b) for $-X$ in combination with Lemma 1 (illumination cap) imply $I(K) \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}/(Cn\log n) = \frac{c'}{\sqrt{n}\log n}\left(\frac{1}{\cos(\pi/14)}\right)^n$.

Glazyrin ($\geq$2023) noted that the base of the exponent $\frac{1}{\cos(\pi/14)} \approx 1.026$ can be improved to $\frac{1}{4}\sqrt{\frac{1}{6}(111 - \sqrt{33})} \approx 1.047$ by a slight modification of the construction: choosing the bases of the cones from a concentric sphere of smaller radius.

New lower bound on $g(n)$

Recall that $g(n)$ is the smallest number of balls of diameter < 1 needed to cover an arbitrary set of diameter 1 in $\mathbb{E}^n$.

Bourgain and Lindenstrauss (1989): $g(n) \geq 1.0645^n$

Theorem 2

$$g(n) \geq \frac{c}{\sqrt{n} \log n} \left(\frac{2}{\sqrt{3}} \right)^n \qquad \textit{(note that } \frac{2}{\sqrt{3}} \approx 1.1547\textit{)}$$

Proof: Use Lemma 3 with $\varphi = \frac{\pi}{3}$ to get a thinly spread $X \subset \mathbb{S}^{n-1}$.
By Lemma 2 (i) (separation lemma) with $\alpha = \frac{\pi}{6}$, $\operatorname{diam} X \leq 2 \cos \frac{\pi}{6} = \sqrt{3}$.
Any ball of diameter $\sqrt{3}$ intersects $\mathbb{S}^{n-1}$ by a cap of radius $\leq \varphi$, so by Lemma 3 (b) we need at least $\frac{c\sqrt{n}}{(\sin \varphi)^n} / (Cn \log n) = \frac{c'}{\sqrt{n} \log n} \left(\frac{2}{\sqrt{3}} \right)^n$ such caps to cover X.

Thinly spread subsets of the sphere

Denote $\mu(\varphi) := \mu(C(x,\varphi))$, $x \in \mathbb{S}^{n-1}$.

Theorem 3

There is n_0 such that for any $n \geq n_0$, $\psi \in (0, \frac{\pi}{2})$ and $\varphi \in (\frac{1}{n}, \frac{\pi}{2})$ there exists a collection $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ with $N \geq \min\{\frac{4n \log n}{\mu(\varphi)}, \frac{1}{8\mu(\psi)}\}$ such that

(a) $\psi \leq \theta(x_i, x_j) \leq \pi - \psi$ *for all* $i \neq j$;

(b) $|\{i : x \in C(x_i, \varphi)\}| \leq 400 n \log n$ *for all* $x \in \mathbb{S}^{n-1}$.

Lemma 3 is obtained when $\psi = \varphi$.

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Proof outline: Let Y be a set of $M = \lceil \frac{8n \log n}{\mu((1-\frac{1}{2n})\varphi)} \rceil$ uniformly i.i.d. points from $\mathbb{S}^{n-1}$. By Böröczky and Wintsche (2003), Y satisfies (b) w.h.p.

- For $U \subset Y$, let $B(U) := \{\{u, v\} : \theta(u, v) \notin [\psi, \pi - \psi], u, v \in U, u \neq v\}$.

A pair of points from Y is in $B(Y)$ with probability $p = 2\mu(\psi)$.
Thus $\mathbb{E}(|B(Y)|) \leq p\frac{M^2}{2}$ and $\exists$ Y satisfying (b) with $|B(Y)| < pM^2$.

- If $pM \leq \frac{1}{2}$, then $|B(Y)| < \frac{M}{2}$, and a point from each pair in $B(Y)$ can be removed to obtain the desired $X \subset Y$ with $N \geq \frac{M}{2} \geq \frac{4n \log n}{\mu(\varphi)}$.
- If $pM > \frac{1}{2}$, draw $T \subset Y$ selecting each point with probability $\frac{1}{2pM}$.

Then $\mathbb{E}(|T| - |B(T)|) \geq \frac{1}{2p} - pM^2(\frac{1}{2pM})^2 = \frac{1}{4p} = \frac{1}{8\mu(\psi)}$.

Illumination of convex bodies close to ball

For $D > 1$ let $\mathcal{K}_D^n$ be the family of all convex bodies K in $\mathbb{E}^n$ such that

$$\mathbb{B}^n \subset K \subset D\mathbb{B}^n.$$

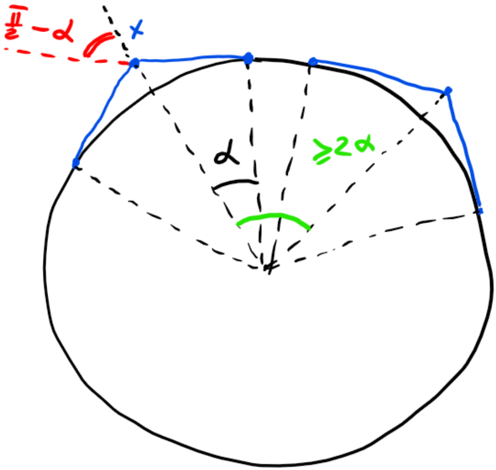
Naszódi (2016): for any fixed $1 < D < 1.116$ and sufficiently large n

$$\frac{1}{20} D^n \leq \sup_{K \in \mathcal{K}_D^n} I(K) \leq (cn^{3/2} \log n) D^n,$$

where the upper bound is valid for any $D > 1$.

Construction: convex hull of a discrete subset of $D\mathbb{S}^{n-1}$ and $\mathbb{B}^n$.

Spiky ball



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Theorem 4

For any fixed $1 < D < \frac{2}{\sqrt{3}}$ (≈ 1.1547) *and sufficiently large* n

$$c\sqrt{n} D^n \leq \sup_{K \in \mathcal{K}_D^n} I(K).$$

Illumination of bodies of constant width close to the ball

For $D > 1$ let $\mathcal{W}_D^n$ be the family of all convex bodies of *constant width* $K \subset \mathbb{E}^n$ such that

$$\mathbb{B}^n \subset K \subset D\mathbb{B}^n.$$

Theorem 5

For any fixed $1 < D < \frac{1}{2\cos(\pi/14)-1} \quad (\approx 1.0528)$ *and sufficiently large* n

$$c\sqrt{n}\left(\frac{2D}{D+1}\right)^n \leq \sup_{K \in \mathcal{W}_D^n} I(K) \leq (Cn^{3/2}\log n)\left(\frac{2D}{D+1}\right)^n,$$

where the upper bound is valid for any $D > 1$.

Covering by balls of smaller diameter

For $K \subset \mathcal{W}_D^n$ of width w let $g(K)$ denote the smallest number of balls of diameter less than w needed to cover K.

Theorem 6

For any fixed $1 < D < \frac{1}{\sqrt{3}-1}$ (≈ 1.366) *and sufficiently large* n

$$c\sqrt{n}\left(\frac{2D}{D+1}\right)^n \leq \sup_{K\in\mathcal{W}_D^n} g(K) \leq (Cn^{3/2}\log n)\left(\frac{2D}{D+1}\right)^n,$$

where the upper bound is valid for any $D > 1$.

Concluding remarks

Upper bounds in the last two theorems are achieved in a "universal" way: illumination directions and covering balls do not depend on K, only on D.

Our constructions of bodies of constant width also provide the same exponential lower bounds for "mix and match" covering by balls of smaller diameter and smaller homothets.

Question

Can $b(n) \leq (\sqrt{3/2} + o(1))^n$ be improved using "mix and match" covering by balls of smaller diameters and smaller homothets?

$I(K)$ and $g(K)$ have the same order for $K \in \mathcal{W}_D^n$ when D is close to 1.

Question

Is it true that $I(K) = g(K)$ for any K of constant width? If not, are $I(K)$ and $g(K)$ for constant width $K \subset \mathbb{E}^n$ equivalent up to a factor polynomial in n?