# On behaviour of Christoffel function for convex domains

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For a compact set $D \subset \mathbb{R}^d$ with non-empty interior and a positive weight function $w \in L_1(D)$, the associated Christoffel function is defined as

$$\lambda_n(D, w, \mathbf{x}) = \left(\sum_{k=1}^{N} \varphi_k(\mathbf{x})^2\right)^{-1}, \quad \mathbf{x} \in D,$$

where $\mathcal{P}_n = \mathcal{P}_{n,d}$ is the space of all real algebraic polynomials of total degree $\leq n$ in $d$ variables, and $\{\varphi_k\}_{k=1}^N$ is an orthonormal basis of $\mathcal{P}_n$ w.r.t. $\langle f, g\rangle = \int_D f(\mathbf{y})g(\mathbf{y})w(\mathbf{y})d\mathbf{y}$.

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Equivalently, the Christoffel function can be defined through the following extremal property:

$$\lambda_n(D, w, \mathbf{x}) = \min_{f \in \mathcal{P}_n,\, f(\mathbf{x})=1} \int_D f^2(\mathbf{y}) w(\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in D.$$

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If $w \equiv 1$ is the uniform weight, we will write $\lambda_n(D, \mathbf{x}) = \lambda_n(D, w, \mathbf{x})$.

**Th. (Bos, Della Vecchia, Mastroianni, 1998)** *For centrally symmetric positive continuous weight $w$ on the unit ball in $\mathbb{R}^d$ one has*

$$\lim_{n\to\infty} \lambda_n(B^d, w, \mathbf{x}) \binom{n+d}{d} = \frac{\pi^{\frac{d+1}{2}} w(\mathbf{x}) \sqrt{1-|\mathbf{x}|^2}}{\Gamma(\frac{d+1}{2})}, \quad |\mathbf{x}| < 1,$$

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This is an example of a typical result on computation of asymptotics of Christoffel function, which usually establishes that

$$\lim_{n\to\infty} n^d \lambda_n(D, w, \mathbf{x}) = \rho(D, w, \mathbf{x})$$

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It is crucial to know structure of orthonormal polynomials for results of this type. Unfortunately, we know almost nothing for domains admitting any generality.

**Th. (Kroo, 2015)** *If $K \subset \mathbb{R}^d$ is a starlike $C^\alpha$ domain, $0 < \alpha \le 2$, $w$ is positive and continuous on $K$, then*

$$\liminf_{n\to\infty} n^d \lambda_n(K, w, \mathbf{x}) \ge c(K, \alpha) w(\mathbf{x})(1 - \varphi_K(\mathbf{x}))^{\gamma(\alpha,d)}, \quad x \in K \setminus \partial K,$$

*where*

$$\varphi_K(\mathbf{x}) := \inf\left\{\mu > 0 : \frac{\mathbf{x}}{\mu} \in K\right\} \quad \textit{and} \quad \gamma(\alpha, d) = \frac{1}{2} + \frac{(d-1)(2-\alpha)}{2\alpha}.$$

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**Th. (Kroo, 2015)** *If* $1 < \alpha < 2$*, for any* $\mathbf{z} = (t, 0, \ldots, 0) \in B^d_\alpha$*,* $t \in [1/2, 1]$*,*

$$\limsup_{n\to\infty} \left(\frac{n}{\log n}\right)^d \lambda_n(B^d_\alpha, w, \mathbf{z}) \le c(d,\alpha) w(\mathbf{z})(1-\varphi_K(\mathbf{z}))^{\gamma(\alpha,d)},$$

*where*

$$B^d_\alpha := \{(x_1, \ldots, x_d) : |x_1|^\alpha + \cdots + |x_d|^\alpha \le 1\}.$$

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$$\frac{1}{c}\int_{I_x} w(t)dt \le \lambda_n([-1,1], w, x) \le c\int_{I_x} w(t)dt,$$

where

$$I_x = [x-\rho_n(x), x+\rho_n(x)] \cap [-1,1], \quad \rho_n(x) = n^{-2} + n^{-1}\sqrt{1-x^2},$$

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Bounds on asymptotics for uniform weight transfer to any positive continuous weight by universality in the bulk results of **Kroo and Lubinsky (2013)**.

If $D$ is a starlike body in $\mathbb{R}^d$, then for any point $\mathbf{x} \in D$

$$\lambda_n(D, \mathbf{x}) \approx \lambda_n(D, \mu\mathbf{x}), \quad \mu \in [1 - c(d)n^{-2}, 1],$$

where $c(d) = 2^{-3-d/2}$ (constants in "$\approx$" are absolute).

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Behaviour for the ball:

$$\lambda_n(B^d, \mathbf{x}) \approx c(d, \sigma)n^{-d}\sqrt{1 - |\mathbf{x}|^2}, \quad \mathbf{x} \in (1 - \sigma n^{-2})B^d.$$

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In particular,

$$\min_{\mathbf{x} \in B^d} \lambda_n(B^d, \mathbf{x}) \approx c(d)n^{-(d+1)}.$$

Recall that

$$\lambda_n(D, \mathbf{x}) = \min_{f \in \mathcal{P}_n,\, f(\mathbf{x})=1} \int_D f^2(\mathbf{y}) d\mathbf{y}, \quad \mathbf{x} \in D.$$

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If $D_1 \subset D_2 \subset \mathbb{R}^d$ then

$$\lambda_n(D_1, \mathbf{x}) \le \lambda_n(D_2, \mathbf{x}), \quad \mathbf{x} \in D_2. \quad \text{(comparison)}$$

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For an affine transform $T\mathbf{x} = \mathbf{x}_0 + A\mathbf{x}$ of $\mathbb{R}^d$, where $\mathbf{x}_0 \in \mathbb{R}^d$ and $A$ is an $n \times n$ matrix, we will write $\det T = \det A$ and usually assume $\det T \neq 0$. We have

$$\lambda_n(TD, T\mathbf{x}) = \lambda_n(D, \mathbf{x}) |\det T|, \quad \mathbf{x} \in D. \quad \text{(affine transform)}$$

**Th. (Ditzian, P., 2016)** *Let $D \subset \mathbb{R}^d$ be a compact set, $\mathbf{y} = (y_1, \dots, y_d) \in [-1,1]^d$, $T$ be an affine transformation of $\mathbb{R}^d$ such that $D \subset T([-1,1]^d)$ and $T\mathbf{y} \in D$. Then*

$$\lambda_n(D, T\mathbf{y}) \le c(d) |\det T| \rho_n(y_1)\rho_n(y_2)\dots\rho_n(y_d).$$

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To deduce

$$\lambda_n(B^d, \mathbf{x}) \le c(d,\sigma) n^{-d}\sqrt{1-|\mathbf{x}|^2}, \quad \mathbf{x} \in (1-\sigma n^{-2})B^d,$$

choose $T$ as identity and note rotation invariance, $\rho_n(0) = n^{-1}$, and that $\rho_n(x) \le c(\sigma) n^{-1}\sqrt{1-x^2}$ for $|x| < 1 - \sigma n^{-2}$.

In the other direction, the difficulty in showing

$$\lambda_n(B^d, \mathsf{x}) \geq c(d, \sigma) n^{-d} \sqrt{1 - |\mathsf{x}|^2}, \quad \mathsf{x} \in (1 - \sigma n^{-2}) B^d,$$

is that for extremal polynomial $P \in \mathcal{P}_{n,d}$ satisfying $P(\mathsf{x}) = 1$ and $\|P\|^2_{L_2(B^d)} = \lambda_n(B^d, \mathsf{x})$ we do not know if $\|P\|^2_{L_\infty(B^d)} = 1$.

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is that for extremal polynomial $P \in \mathcal{P}_{n,d}$ satisfying $P(\mathbf{x}) = 1$ and $\|P\|^2_{L_2(B^d)} = \lambda_n(B^d, \mathbf{x})$ we do not know if $\|P\|^2_{L_\infty(B^d)} = 1$. Yet, it is possible to show $\|P\|_{L_\infty((1-\delta/2)B^d)} \leq c(d)$ and then use Bernstein(-Markov)-type inequalities to find a neighborhood of $\mathbf{x}$ of measure $c(d,\sigma) n^{-d} \sqrt{1-|\mathbf{x}|^2}$ in which $P \geq \frac{1}{2}$.

**Lemma. (P., ≥2017)** *Suppose a convex body $D \subset \mathbb{R}^d$ is contained in a ball of radius $R$. For any $\mathbf{x} \in D \setminus \partial D$, let $\mathbf{u} \in \mathbb{R}^d$ be a unit vector such that*

$$\delta := \max\{t : \mathbf{x} + t\mathbf{u} \in D\} \leq \nu \operatorname{dist}(\mathbf{x}, \partial D)$$

*for some $\nu \geq 1$. If $\delta \geq \sigma n^{-2}$, $\sigma > 0$, then*

$$\lambda_n(D, \mathbf{x}) \leq c(R, d, \nu, \sigma) n^{-d} \sqrt{\delta}.$$

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A similar domain-independent bound in the other direction was proved by **Kroo (2015)** with the main term $n^{-d}\delta^{d/2}$.

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**Th. (P., ≥2017)** *Suppose a planar convex body $D$ is contained in a disc of radius $R$, and for some $\mathbf{x} \in D \setminus \partial D$ and unit vector $\mathbf{u} \in \mathbb{R}^d$ there are $r > 0$ and $t_0 < 0$ such that $rB^2 + \mathbf{x} + t_0\mathbf{u} \subset D$. Let $\delta := \max\{t : \mathbf{x} + t\mathbf{u} \in D\}$ and $l_i := \max\{t : \mathbf{x} + (-1)^i t\mathbf{v} \in D\}$, $i = 1, 2$, where $\mathbf{v}$ is one of the two unit vectors orthogonal to $\mathbf{u}$. If $\delta \geq \sigma n^{-2}$, $\sigma > 0$, then*

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Corollary: extra $(\log n)^2$ term can be removed in Kroo's upper bound on $\lambda_n(B^2_\alpha, \mathbf{z})$.

**Th. (P., Usoltseva, ≥2017)** *Suppose $D$ is a planar convex body, $\mathbf{u} \in \mathbb{R}^d$ is a unit vector, and $\mathbf{y} \in \partial D$ is such that $\mathbf{y} - \beta\mathbf{u} \in D$. Let*

$$l_i(\delta) := \max\{t : \mathbf{y} - \delta\mathbf{u} + (-1)^i t\mathbf{v} \in D\}, \quad i = 1, 2, \quad 0 < \delta < \beta,$$

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**Corollary. (P., Usoltseva, $\geq$2017)** *Let $\mathbf{y} \in \partial B^2_\alpha$, $1 < \alpha < 2$, $0 < y_1 \leq y_2$, $\mathbf{u}$ be the outward unit normal at $\mathbf{y}$, $\sigma n^{-2} < \delta < 1$, $\sigma > 0$. Then*

$$\lambda_n(B^2_\alpha, \mathbf{y} - \delta\mathbf{u}) \approx c(\alpha, \sigma) n^{-2} \delta^{\frac{1}{2}} (\max\{\delta, y_1^\alpha\})^{\frac{1}{\alpha} - \frac{1}{2}}$$

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$$\lambda_n(D, \mathbf{x}) \leq c(r, R, \sigma) n^{-2} \sqrt{\min\{l_1 l_2, \delta\}}.$$

This upper bound cannot be improved if only $l_1$, $l_2$, $\delta$ are measured.

**Th. (P., $\geq$2017)** *Suppose a planar convex body $D$ is contained in a disc of radius $R$, and for some $\mathbf{x} \in D \setminus \partial D$ and unit vector $\mathbf{u} \in \mathbb{R}^d$ there are $r > 0$ and $t_0 < 0$ such that $rB^2 + \mathbf{x} + t_0\mathbf{u} \subset D$. Let $\delta := \max\{t : \mathbf{x} + t\mathbf{u} \in D\}$ and $l_i := \max\{t : \mathbf{x} + (-1)^i t\mathbf{v} \in D\}$, $i = 1, 2$, where $\mathbf{v}$ is one of the two unit vectors orthogonal to $\mathbf{u}$. If $\delta \geq \sigma n^{-2}$, $\sigma > 0$, then*

$$\lambda_n(D, \mathbf{x}) \leq c(r, R, \sigma) n^{-2} \sqrt{\min\{l_1 l_2, \delta\}}.$$

This upper bound cannot be improved if only $l_1$, $l_2$, $\delta$ are measured.

**Th. (P., $\geq$2017)** *For any positive $l_1$, $l_2$, $\delta$ with $10\delta < l_1, l_2 < \frac{1}{10}$, one can find a planar convex body $D \subset 4B^2$ and $\mathbf{x} \in D$ satisfying for $\mathbf{u} := \mathbf{x}/|\mathbf{x}|$ that $B^2 = B^2 + \mathbf{x} - |\mathbf{x}|\mathbf{u} \subset D$, $\delta = \max\{t : \mathbf{x} + t\mathbf{u} \in D\}$, $l_i = \max\{t : \mathbf{x} + (-1)^i t\mathbf{v} \in D\}$, $i = 1, 2$, where $\mathbf{v}$ is one of the two unit vectors orthogonal to $\mathbf{u}$, and that for any $n$ with $\delta > \sigma n^{-2}$, $\sigma > 0$, the following inequality holds:*

$$\lambda_n(D, \mathbf{x}) \geq c(\sigma) n^{-2} \sqrt{\min\{l_1 l_2, \delta\}}.$$

**Th. (P., $\geq$2017)** *Suppose a convex body $D \subset \mathbb{R}^d$ contains a ball of radius $r$ and is contained in a ball of radius $R$. For any $\mathbf{x} \in D \setminus \partial D$, let $\mathbf{u} \in \mathbb{R}^d$ be a unit vector such that for some $\nu \geq 1$*

$$\delta := \max\{t : \mathbf{x} + t\mathbf{u} \in D\} \leq \nu \mathrm{dist}(\mathbf{x}, \partial D)$$

*and the hyperplane passing through $\mathbf{x}+\delta\mathbf{u}$ with normal vector $\mathbf{u}$ is supporting to $D$. If $\delta \geq \sigma n^{-2}$, $\sigma > 0$, then*

$$\lambda_n(D, \mathbf{x}) \leq c(d, r, R, \nu, \sigma) n^{-d} \min\{\sqrt{\delta}, \delta^{1-\frac{d}{2}} \mathrm{Vol}_{d-1}(\{\mathbf{y} \in D : (\mathbf{x}-\mathbf{y}) \perp \mathbf{u}\})\}.$$

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Idea: comparison with a properly chosen parallelotope containing $D$ defined using simplex of largest $(d-1)$ volume contained in the hyperplane section $\{\mathbf{y} \in D : (\mathbf{x} - \mathbf{y}) \perp \mathbf{u}\}$.

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**Th. (P., ≥2017)** *Suppose a convex body $D \subset \mathbb{R}^d$ contains a ball of radius $r$ and is contained in a ball of radius $R$. For any $\mathbf{x} \in D \setminus \partial D$, let $\mathbf{u} \in \mathbb{R}^d$ be a unit vector such that for some $\nu \geq 1$*

$$\delta := \max\{t : \mathbf{x} + t\mathbf{u} \in D\} \leq \nu \mathrm{dist}(\mathbf{x}, \partial D)$$

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$$\lambda_n(D, \mathbf{x}) \leq c(d, r, R, \nu, \sigma) n^{-d} \min\{\sqrt{\delta}, \delta^{1-\frac{d}{2}} \mathrm{Vol}_{d-1}(\{\mathbf{y} \in D : (\mathbf{x}-\mathbf{y}) \perp \mathbf{u}\})\}.$$

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Cannot be improved if only $\delta$ and $\mathrm{Vol}_{d-1}(\{\mathbf{y} \in D : (\mathbf{x} - \mathbf{y}) \perp \mathbf{u}\})$ are used.

Corollary: extra $(\log n)^d$ term can be removed in Kroo's upper bound on $\lambda_n(B^d_\alpha, \mathbf{z})$.

Nikol'skii inequality: for $D \subset \mathbb{R}^d$ and $0 < p \le q \le \infty$ with $\mu = \mu(D) > 0$

$$\|P\|_{L_q(D)} \le cn^{(\frac{1}{p} - \frac{1}{q})\mu} \|P\|_{L_p(D)}, \quad P \in \mathcal{P}_{n,d}.$$

For example, $\mu([-1,1]^d) = 2d$.

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If $\min_{x \in D} \lambda_n(D, \mathbf{x}) \approx c(D) n^{-\mu}$, then the above Nikol'skii inequality holds with this $\mu$.

$\mu(B^d) = d + 1$
$\mu(B^d_\alpha) = d + 1$, $2 \le \alpha < \infty$
$\mu(B^d_\alpha) = 2 + \frac{2}{\alpha}(d-1)$, $1 < \alpha < 2$ **(Ditzian, P., 2016)**
$\mu(B^d_1) = \mu(B^d_\infty) = 2d$

Half-ball $B^3_+ = \{(x_1, x_2, x_3) \in B^3 : x_3 \geq 0\}$ does not properly fall into $C^\alpha$ classification of Kroo.

Half-ball $B^3_+ = \{(x_1, x_2, x_3) \in B^3 : x_3 \geq 0\}$ does not properly fall into $C^\alpha$ classification of Kroo.

$\mu(B^3_+) = 5$ **(Ditzian, P., 2016)**

$\lambda_n(B^3_+, (1-\delta, 0, \delta/4)) \approx c(\sigma) n^{-3}\delta$, $\sigma n^{-2} < \delta < 1/3$, $\sigma > 0$ **(P., $\geq$ 2017)**
The hyperplane section for upper bound is anisotropic.

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