On behaviour of Christoffel function for domains in $\mathbb{R}^d$

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Christoffel function and Nikol'skii inequality

For a bounded domain $D \subset \mathbb{R}^d$ and a finite dimensional space $\mathcal{S}$ of bounded continuous functions on D, the Christoffel function is given by

$$\lambda(\mathcal{S}, D, \mathbf{x})^{-1} = C(\mathcal{S}, D, \mathbf{x}) = \sum_{k=1}^{N} \varphi_k(\mathbf{x})^2, \quad \mathbf{x} \in D,$$

where $\{\varphi_k\}_{k=1}^N$ is an orthonormal basis of $\mathcal{S}$ on D with respect to the Lebesgue measure on D.

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Nikol'skii inequality: for $D \subset \mathbb{R}^d$ and $0 < p \le q \le \infty$ with $\sigma = \sigma(D) > 0$

$$\|P\|_{L_q(D)} \le c n^{\sigma(\frac{1}{p} - \frac{1}{q})} \|P\|_{L_p(D)}, \quad P \in \mathcal{P}_{n,d},$$

where $\mathcal{P}_{n,d}$ are algebraic polynomials of total degree $\le n$ in d variables.

Equivalent variational definition of Christoffel function:

$$C(\mathcal{S}, D, \mathbf{x}) = \max_{P \in \mathcal{S},\, P(\mathbf{x})=1} \|P\|_{L_2(D)}^{-2}.$$

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In our context: If $C(\mathcal{P}_{n,d}, D) := \sup_{\mathbf{x} \in D} C(\mathcal{P}_{n,d}, D, \mathbf{x}) \approx n^{\sigma}$,
then for $0 < p \leq q \leq \infty$

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Some examples:
$\sigma([-1,1]^d) = 2d$ (cube),
$\sigma(B_2^d) = d + 1$ (Euclidean ball).

Studies of asymptotics of Christoffel function usually focus on:
behaviour of $C(\mathcal{P}_{n,d}, D, \mathbf{x})$ when $\mathbf{x} \in D$ is fixed and then $n \to \infty$ with typical results of the form $\lim_{n\to\infty} n^{-\mu} C(\mathcal{P}_{n,d}, D, \mathbf{x}) = \Phi(\mathbf{x})$.

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Applications to Nikol'skii inequality require:
behaviour of $C(\mathcal{P}_{n,d}, D) = \sup_{\mathbf{x}\in D} C(\mathcal{P}_{n,d}, D, \mathbf{x})$, so n is fixed first, the supremum over $\mathbf{x} \in D$ is taken, and then $n \to \infty$.
Typical result $C(\mathcal{P}_{n,d}, D) \approx n^{\sigma}$.

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For lower bounds we use the variational definition and give constructions of polynomials (based on Chebyshev polynomials) that provide matching bounds in many situations.

Comparison between domains:
if $D_1 \subset D_2 \subset \mathbb{R}^d$ are two bounded domains, then

$$C(\mathcal{P}_{n,d}, D_1, \mathsf{x}) \geq C(\mathcal{P}_{n,d}, D_2, \mathsf{x}), \quad \mathsf{x} \in D_1 .$$

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Affine transforms:
for any non-degenerate affine transform T ($T\mathbf{x} = \mathbf{x}_0 + A\mathbf{x}$, $\det T := \det A$) on $\mathbb{R}^d$ and any bounded domain $D \subset \mathbb{R}^d$, we have

$$C(\mathcal{P}_{n,d}, T(D), T\mathbf{x}) = C(\mathcal{P}_{n,d}, D, \mathbf{x}) |\det T|^{-1}, \quad \mathbf{x} \in D.$$

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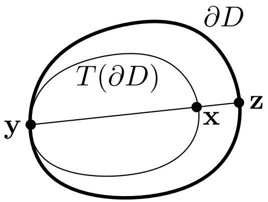
Observation: if $x \in D$, then for some affine transform T with $|\det T| \ge \frac{1}{2^d}$, we have

$$\mathbf{x} \in T(\partial D) \quad \text{and} \quad T(D) \subset D.$$



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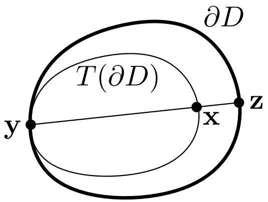

Conjecture: if $D \subset \mathbb{R}^d$ is a convex body, then for each n, $\max_{\mathbf{x} \in D} C(\mathcal{P}_{n,d}, D, \mathbf{x})$ is attained at an extreme point of D.

Euclidean ball: $C(\mathcal{P}_{n,d}, B_2^d) = C(\mathcal{P}_{n,d}, B_2^d, (1, 0, \dots, 0)) \approx n^{d+1}$.

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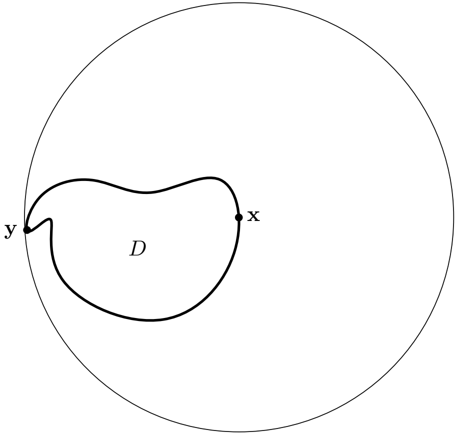
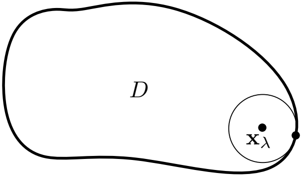


Matching upper bound: if $D \subset \mathbb{R}^d$ is a compact set, and $D = \cup_\lambda(\mathbf{x}_\lambda + r_\lambda B_2^d)$ with $\inf_\lambda r_\lambda > 0$, then $C(\mathcal{P}_{n,d}, D) \leq cn^{d+1}$ (so $C(\mathcal{P}_{n,d}, D) \approx n^{d+1}$).



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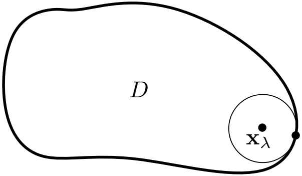

This is true if D has sufficiently smooth ∂D: in particular, when ∂D is C^2, and more generally, when ∂D is C^1 and unit normal satisfies Lipschitz condition.

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Lower bound: if $D \subset \mathbb{R}^d$ is a compact set, T is an affine transform with $D \subset T(Q)$ and $\mathbf{x} := T(1, \ldots, 1) \in D$, then

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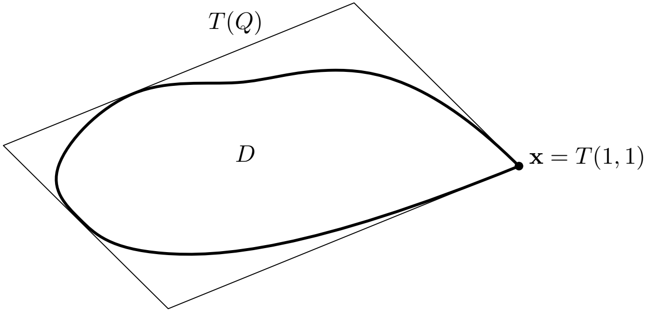
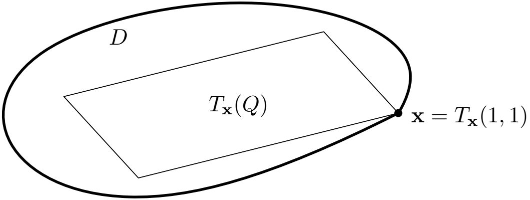


Upper bound: if $D \subset \mathbb{R}^d$ is a compact set and for any $\mathbf{x} \in D$ there is $T_{\mathbf{x}}$ satisfying $\mathbf{x} = T_{\mathbf{x}}(1, \ldots, 1)$, $T_{\mathbf{x}}(Q) \subset D$, while $\inf_{\mathbf{x}} |\det T_{\mathbf{x}}| > 0$, then $C(\mathcal{P}_{n,d}, D) \le cn^{2d}$.



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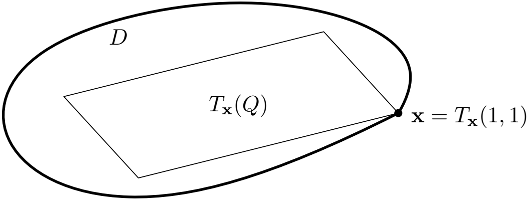

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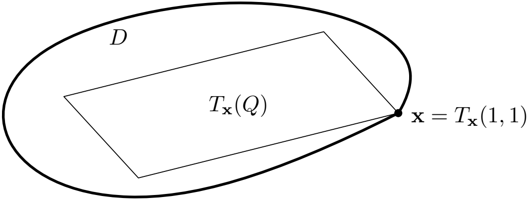

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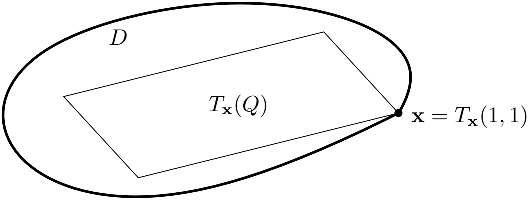

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If D has sufficiently smooth ∂D (e.g. ∂D is C^2), then $C(\mathcal{P}_{n,d}, D) \approx n^{d+1}$.

Model examples:

$$B_\alpha^d := \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_\alpha = \left(|x_1|^\alpha + \cdots + |x_d|^\alpha\right)^{\frac{1}{\alpha}} \leq 1\}, \quad 1 \leq \alpha < \infty$$

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So far, we obtained that $C(\mathcal{P}_{n,d}, B_\alpha^d) \approx \begin{cases} n^{2d}, & \alpha = 1, \infty, \\ n^{d+1}, & 2 \le \alpha < \infty. \end{cases}$

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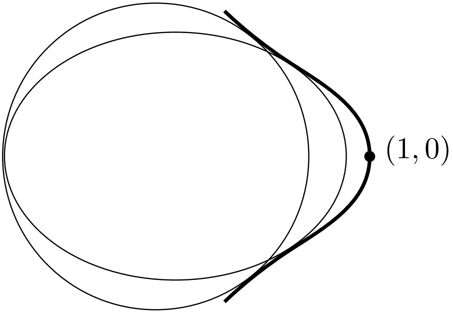

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We can define $C(\mathcal{P}_{n,d}, D, \mathbf{x})$ for any $\mathbf{x} \in \mathbb{R}^d$ by $C(\mathcal{P}_{n,d}, D, \mathbf{x}) = \sum_{k=1}^{N} \varphi_k(\mathbf{x})^2$,

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The following tools and properties transfer to the extended definition:

$$C(\mathcal{P}_{n,d}, D, \mathbf{x}) = \max_{P \in \mathcal{P}_{n,d},\, P(\mathbf{x})=1} \|P\|_{L_2(D)}^{-2}, \ \mathbf{x} \in \mathbb{R}^d,$$

$$D_1 \subset D_2 \subset \mathbb{R}^d \Rightarrow C(\mathcal{P}_{n,d}, D_1, \mathbf{x}) \geq C(\mathcal{P}_{n,d}, D_2, \mathbf{x}), \ \mathbf{x} \in \mathbb{R}^d,$$

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Ellipsoid with "handle"

Define $\mathbf{v}_n^d := (1 + n^{-2}/3, 0, \ldots, 0) \in \mathbb{R}^d$.
Recalling that $C(\mathcal{P}_{n,d}, B_2^d, (1, 0, \ldots, 0)) \approx n^{d+1}$,
Markov's inequality implies $C(\mathcal{P}_{n,d}, B_2^d, \mathbf{v}_n^d) \leq c(d) n^{d+1}$.

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Theorem (main result for upper estimates). If $D \subset \mathbb{R}^d$ is a compact set, T is an affine transform of $\mathbb{R}^d$ with $T(B_2^d) \subset D$, then

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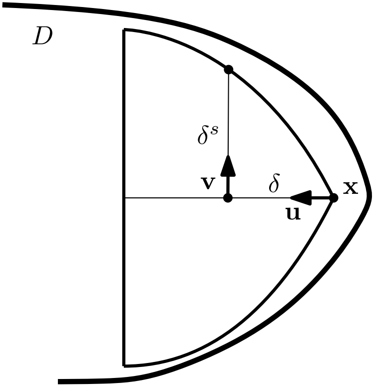
Example: $C(\mathcal{P}_{n,2}, [-1, 1]^2, (\pm 1, \pm 1)) \leq cn^4$.

"Cone"-type sets

Theorem. If $D \subset \mathbb{R}^d$ is a compact set, $\mathbf{x} \in D$, $\mathbf{u} \in \mathbb{R}^d$, $|\mathbf{u}| = 1$, $\beta, \gamma > 0$, $s \in [\frac{1}{2}, 1]$, and

$$\{\mathbf{x} + \delta\mathbf{u} + \lambda\delta^s\mathbf{v} : \delta \in [0, \beta], \lambda \in [0, \gamma], \mathbf{v} \in \mathbb{R}^d, |\mathbf{v}| = 1, \mathbf{u} \perp \mathbf{v}\} \subset D,$$

then $C(\mathcal{P}_{n,d}, D, \mathbf{x}) \le c(d, \beta, \gamma) n^{2+2s(d-1)}$.



$s = 1$: right circular cone, $s = 1/2$: almost a spherical cap.

Corollary: upper estimate on $C(\mathcal{P}_{n,d}, B^d_\alpha)$ for $1 < \alpha < 2$

At any $\mathbf{x} \in \partial B^d_\alpha$ we can inscribe a "cone"-type set with $s = 1/\alpha$, "vertex" at $\mathbf{x}$, and "axis" in the direction opposite to the outward normal vector to the surface ∂B^d_α at $\mathbf{x}$.

$C(\mathcal{P}_{n,d}, B^d_\alpha) \leq c(\alpha, d) n^{2+\frac{2}{\alpha}(d-1)}$, $1 < \alpha < 2$.

Lower estimates: basic univariate construction

Denote $\rho_n(x) := \frac{1}{n^2} + \frac{1}{n}\sqrt{1-x^2}$. For any $n, m \geq 1$ and $y \in [-1, 1]$, there exists a polynomial $P_n = P_{n,m,y}$ of degree $\leq n$ such that $P_n(y) = 1$ and

$$|P_n(x)| \leq c(m) \left(\frac{\rho_n(y)}{\rho_n(y) + |x - y|} \right)^m, \quad x \in [-1, 1].$$

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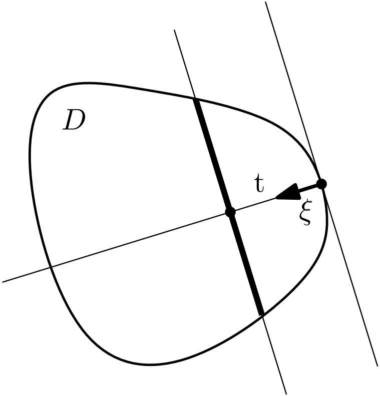
Two ways to obtain multivariate polynomials:
ridge functions and tensor product.

Lower estimate in terms of parallel section function

For a unit vector $\boldsymbol{\xi} \in \mathbb{R}^d$ and a non-empty compact set $D \subset \mathbb{R}^d$,

$$A_{D,\boldsymbol{\xi}}(t) := \mathrm{Vol}_{d-1}(D \cap \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \boldsymbol{\xi} = t + h\}),$$

where $h \in \mathbb{R}$ is the smallest h such that $D \cap \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \boldsymbol{\xi} = t + h\} \neq \emptyset$.



Theorem. If $D \subset \mathbb{R}^d$ is a compact set, $\boldsymbol{\xi} \in \mathbb{R}^d$, $|\boldsymbol{\xi}| = 1$, and $n, m \geq 1$, then

$$C(\mathcal{P}_n, D) \geq c(m, D) \left(\int_0^{n^{-2}} A_{D,\boldsymbol{\xi}}(t)\, dt + n^{-2m} \int_{n^{-2}}^{\infty} A_{D,\boldsymbol{\xi}}(t) t^{-m}\, dt \right)^{-1}.$$

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In particular, if $A_{D,\boldsymbol{\xi}}(t) \leq M t^{\lambda}$, $t > 0$, for some $M > 0$, $\lambda < m - 1$, then

$$C(\mathcal{P}_n, D) \geq c(m, D, M, \lambda) n^{2(1+\lambda)}.$$

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In particular, if $A_{D,\boldsymbol{\xi}}(t) \leq Mt^{\lambda},\ t > 0$, for some $M > 0$, $\lambda < m - 1$, then

$$C(\mathcal{P}_n, D) \geq c(m, D, M, \lambda) n^{2(1+\lambda)}.$$

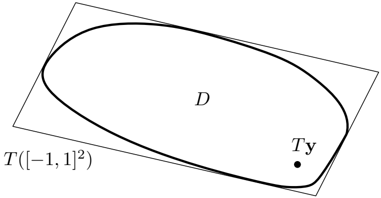
For $D = B_{\alpha}^d$, $1 < \alpha < 2$, we take $\boldsymbol{\xi} = (-1, 0, \dots, 0)$, then $A_{B_{\alpha}^d,\boldsymbol{\xi}}(t) \leq Mt^{(d-1)/\alpha}$, and $C(\mathcal{P}_n, B_{\alpha}^d) \geq cn^{2+\frac{2}{\alpha}(d-1)}$.

So, $C(\mathcal{P}_{n,d}, B_{\alpha}^d) \approx n^{2+\frac{2}{\alpha}(d-1)}$, $1 < \alpha < 2$.

Lower estimate using tensor product and affine transform

Theorem. If $D \subset \mathbb{R}^d$ is a compact set, $\mathbf{y} = (y_1, \dots, y_d) \in [-1,1]^d$, T is an affine transform with $D \subset T([-1,1]^d)$ and $T\mathbf{y} \in D$, then

$$C(\mathcal{P}_{n,d}, D, T\mathbf{y}) \geq c(d) |\det T|^{-1} \rho_n^{-1}(y_1) \rho_n^{-1}(y_2) \dots \rho_n^{-1}(y_d).$$



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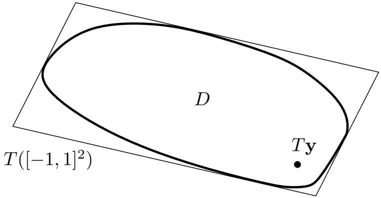
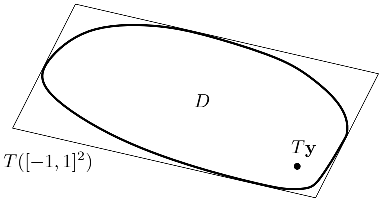


Examples: As $\rho_n(0) \approx n^{-1}$ and $\rho_n(\pm 1) = n^{-2}$, if T is identity, then $C(\mathcal{P}_{n,d}, [-1,1]^d) \geq C(\mathcal{P}_{n,d}, [-1,1]^d, (\pm 1, \dots, \pm 1)) \geq c(d) n^{2d}$, and $C(\mathcal{P}_{n,d}, B_2^d) \geq C(\mathcal{P}_{n,d}, B_2^d, (1, 0, \dots, 0)) \geq c(d) n^{d+1}$.

Question: is it true that for any $\mathbf{x} \in D$

$$C(\mathcal{P}_{n,d}, D, \mathbf{x}) \leq c(D) \sup_{T} |\det T|^{-1} \rho_n^{-1}(y_1)\rho_n^{-1}(y_2)\ldots\rho_n^{-1}(y_d)$$

where the sup is taken over T satisfying $D \subset T([-1,1]^d)$ and $T\mathbf{y} = \mathbf{x}$, for a large class of D, say for convex bodies?



More examples

Half-balls in $\mathbb{R}^d$: $B_+^d := B_2^d \cap \{\mathbf{x} : x_1 \geq 0\}$, $C(\mathcal{P}_{n,d}, B_+^d) \approx n^{d+2}$.

More examples

Half-balls in $\mathbb{R}^d$: $B_+^d := B_2^d \cap \{\mathbf{x} : x_1 \geq 0\}$, $C(\mathcal{P}_{n,d}, B_+^d) \approx n^{d+2}$.

Quarter-ball in $\mathbb{R}^3$: $C(\mathcal{P}_{n,3}, B_+^3 \cap \{\mathbf{x} : x_2 \geq 0\}) \approx n^6$.

σ-s add when Cartesian product is taken,
say for cylinder: $C(\mathcal{P}_{n,3}, B_2^2 \times [-1,1]) \approx n^5$.

Affine transforms affect the constant, but not σ,
say for a "thin" triangle A with vertices $(\pm 1, 0)$ and $(0, \varepsilon)$,
we get $C(\mathcal{P}_{n,2}, A) \approx \varepsilon^{-1} n^4$.

Cohen, Davenport, Leviatan, 2013:

Behaviour of $C(\mathcal{P}_{n,d}, D) = \sup_{\mathbf{x} \in D} C(\mathcal{P}_{n,d}, D, \mathbf{x})$ determines stability and accuracy of least squares approximation.

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Ditzian, $\geq$2016:

Applications of Nikol'skii-type inequalities to relations of coefficients of orthogonal expansion to the norm of the function which contain Hausdorff-Young and Hardy-Littlewood type inequalities.

Corollary for inner points:
If $1 < \alpha < 2$ and $0 < \delta < \frac{1}{2}$, then

$$C(\mathcal{P}_{n,d}, B_\alpha^d, (1-\delta, 0, \dots, 0)) \geq c_0 n^d \delta^{-\gamma(\alpha,d)}, \quad n \geq n_0,$$

where $n_0 = n_0(\delta)$ depends only on δ, $\gamma(\alpha, d) = \frac{1}{2} + \frac{(d-1)(2-\alpha)}{2\alpha}$, and $c_0 > 0$ is an absolute constant.

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In this example, affine transforms of multivariate polynomials obtained through tensor product outperform multivariate fast decreasing polynomials constructed as radial functions.

Computation of Christoffel function

If $D \subset \mathbb{R}^d$ is a simple polytope (d edges out of each vertex), let $H_j : \mathbb{R}^d \to \mathbb{R}$ be the affine functions defining the half-spaces which determine D, i.e.

$$D = \bigcap_j \{\mathbf{x} \in \mathbb{R}^d : H_j(\mathbf{x}) \geq 0\}.$$

Define $l_n(t) := (n^{-2} + n^{-1}\sqrt{t})^{-1}$. Then for any $\mathbf{x} \in \mathcal{D}$ we have

$$C(\mathcal{P}_{n,d}, D, \mathbf{x}) \approx \prod_{j=1}^{d} \ell_j^*(\mathbf{x}),$$

where $\{\ell_j^*(\mathbf{x})\}_{j \geq 1}$ is the non-decreasing rearrangement of $\{l_n(D_j(\mathbf{x}))\}_j$, and the constants of equivalence do not depend on n and $\mathbf{x}$, but depend on D (and choice of H_i).

References:

• A. Cohen, M. Davenport, D. Leviatan, On the Stability and Accuracy of Least Squares Approximations, Found. of Comp. Math., 13 (2013), 819–834.

• Z. Ditzian and A. Prymak, On Nikol'skii inequalities for domains in $\mathbb{R}^d$, Constr. Approx., 44 (2016), 23–51. http://arxiv.org/abs/1409.5397

• A. Kroó, Christoffel functions on convex and starlike domains in $\mathbb{R}^d$, J. Math. Anal. Appl., 421 (2015), no. 1, 718–729.

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• A. Cohen, M. Davenport, D. Leviatan, On the Stability and Accuracy of Least Squares Approximations, Found. of Comp. Math., 13 (2013), 819–834.

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Thank you!