Discrete d-dimensional moduli of smoothness

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joint work with Zeev Ditzian

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For $\boldsymbol{x}, \boldsymbol{h} \in \mathbb{R}^d$, we define $\Delta_{\boldsymbol{h}}^r f(\boldsymbol{x}) := \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(\boldsymbol{x} + k\boldsymbol{h})$.

Denote $I := [0, 1]$, and $\| \cdot \| := \| \cdot \|_{C(I^d)}$.

For $f \in C(I^d)$, the r-th *regular* modulus of smoothness is given by

$$\omega^r(f,t) := \max_{i,\boldsymbol{x},\boldsymbol{h}} \left\{ \left| \Delta_{\boldsymbol{h}}^r f(\boldsymbol{x}) \right| : |\boldsymbol{h}| \le t,\ \boldsymbol{x}, \boldsymbol{x} + r\boldsymbol{h} \in I^d \right\}.$$

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Restricting forward differences to directions parallel to the axes, we obtain the *directional* modulus

$$\overline{\omega}^r(f,t) := \max_{i,\boldsymbol{x},\lambda} \left\{ \left| \Delta_{\lambda \boldsymbol{e}_i}^r f(\boldsymbol{x}) \right| : \lambda \in [0,t],\ \boldsymbol{x}, \boldsymbol{x} + r\lambda\boldsymbol{e}_i \in I^d \right\},$$

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The corresponding *discrete* modulus of smoothness is

$$\Psi_r(f,n) := \max_{i,\boldsymbol{k}} \left\{ \left| \Delta_{2^{-n}\boldsymbol{e}_i}^r f(2^{-n}\boldsymbol{k}) \right| : 2^{-n}\boldsymbol{k}, 2^{-n}(\boldsymbol{k} + r\boldsymbol{e}_i) \in I^d \right\},$$

where $\boldsymbol{k} = (k_1, \ldots, k_d)$ has integer components.

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Clearly, $\quad \Psi_r(f,n) \le \overline{\omega}^r(f, 2^{-n}) \le \omega^r(f, 2^{-n})$.

Theorem. Suppose $f \in C(I^d)$, and let $\Psi_r(n) := \Psi_r(f,n)$. Then

$$\overline{\omega}^r(f, 2^{-n}) \le \overline{M}(r,d) \sum_{k=0}^{\infty} \Psi_r(n+k),$$

$$\omega^r(f, 2^{-n}) \le M(r,d) \Big(\sum_{k=0}^{\infty} \Psi_r(n+k) + \sum_{k=1}^{n_0} 2^{kr} \Psi_r(n-k) + 2^{-nr} \|f\| \Big),$$

where n_0 is the largest integer satisfying $r 2^{n_0 - n - 1} \le 1$, and

$$\omega^1(f, 2^{-n}) \le M' \sum_{k=0}^{\infty} \Psi_1(n+k),$$

where $\overline{M}(r,d)$, $M(r,d)$ and M' are independent of n and f.

Corollaries. (For $f \in C(I^d)$, $\Psi_r(n) := \Psi_r(f, n)$.)

If $\Psi_r(n) = o(2^{-nr})$, then $\Psi_r(n) = 0$, $\overline{\omega}^r(f, t) = 0$, and $f \in \mathcal{P}_{r,d}$ (polynomials of degree $< r$ in each variable).

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If $\Psi_r(n+1) < \lambda \Psi_r(n)$ for some $\lambda < 1$ and all n,
then $\overline{\omega}^r(f, 2^{-n}) \le c\Psi_r(n)$.
If we also have $\Psi_r(n-1) \le \mu \Psi_r(n)$ with $\mu < 2^r$,
then $\omega^r(f, 2^{-n}) \le c\Psi_r(n)$.

Previously known partial cases:

Ciesielski (1977, 1978): certain classes of functions

Ditzian (1987): $d = 1$

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In the current work, our main idea is to use certain interpolatory tensor product splines S_n (continuous functions that are locally in $\mathcal{P}_{r,d}$) and bound $\|f - S_n\|$ using estimates on $\|S_n - S_{n+1}\|$ in terms of $\Psi_r(n + 1)$.

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We need the following algebraic fact: $(r-1) \times (r-1)$ matrix with entries $a_{i,j} = \binom{r}{2j-i}$, $i, j = 1, \ldots, r-1$, is non-degenrate (we assume $\binom{r}{k} = 0$ when $k \notin [0, r]$).

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Ratlief and Rush (1999): $\det\bigl(\{a_{i,j}\}\bigr) = 2^{r(r-1)/2}$.

Remarks.

One can replace I^d by $\mathbb{R}^d$ or $\mathbb{R}_+^d$ in our Theorem.

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One cannot expect to derive bounds on moduli of smoothness in L_p from the values on diadic mesh only.

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