

On Nikol'skii inequalities for domains in \mathbb{R}^d

Andriy Prymak

joint work with Zeev Ditzian

Classical examples of Nikol'skii inequalities: if $0 < p \leq q \leq \infty$, then

$$\|P\|_{L_q[-1,1]} \leq cn^{2(\frac{1}{p}-\frac{1}{q})} \|P\|_{L_p[-1,1]}, \quad P \in \text{span}\{1, \dots, x^{n-1}\},$$
$$\|\tau\|_{L_q[-\pi,\pi]} \leq cn^{(\frac{1}{p}-\frac{1}{q})} \|\tau\|_{L_p[-\pi,\pi]}, \quad \tau \in \text{span}\{e^{ikx} : |k| < n\}.$$

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Our main interest: for $D \subset \mathbb{R}^d$, determine smallest $\sigma = \sigma(D) > 0$ for which

$$\|P\|_{L_q(D)} \leq cn^{\sigma(\frac{1}{p}-\frac{1}{q})}\|P\|_{L_p(D)}, \quad P \in \mathcal{P}_{n,d},$$

where $0 < p \leq q \leq \infty$ and

$\mathcal{P}_{n,d}$ are algebraic polynomials of total degree $\leq n$ in d variables.

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More specific question:

$$\sigma(B_\alpha^d) = ?, \quad 1 \leq \alpha < \infty, \quad \alpha \neq 2.$$

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for all n , then for any $0 < p \leq q \leq \infty$

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Our goal: determine σ in

$$\sup\{\|P\|_{L_2(D)}^{-2} : \|P\|_{L_\infty(D)} = 1, P \in \mathcal{P}_{n,d}\} \approx n^\sigma.$$

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For a bounded domain $D \subset \mathbb{R}^d$ and a finite dimensional space \mathcal{S} of bounded continuous functions on D , the **Christoffel function** is given by

$$\lambda(\mathcal{S}, D, \mathbf{x})^{-1} = C(\mathcal{S}, D, \mathbf{x}) = \sum_{k=1}^N \varphi_k(\mathbf{x})^2, \quad \mathbf{x} \in D,$$

where $\{\varphi_k\}_{k=1}^N$ is an orthonormal basis of \mathcal{S} on D with respect to the Lebesgue measure on D .

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In our context:

$$\sup\{\|P\|_{L_2(D)}^{-2} : \|P\|_{L_\infty(D)} = 1, P \in \mathcal{P}_{n,d}\} \approx n^\sigma.$$

if and only if

$$C(\mathcal{P}_{n,d}, D) := \sup_{\mathbf{x} \in D} C(\mathcal{P}_{n,d}, D, \mathbf{x}) \approx n^\sigma.$$

Studies of asymptotics of Christoffel function usually focus on:
behaviour of $C(\mathcal{P}_{n,d}, D, \mathbf{x})$ when $\mathbf{x} \in D$ is fixed and then $n \rightarrow \infty$
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Applications to Nikol'skii inequality require:

behaviour of $C(\mathcal{P}_{n,d}, D) = \sup_{\mathbf{x} \in D} C(\mathcal{P}_{n,d}, D, \mathbf{x})$, so n is fixed first, the
supremum over $\mathbf{x} \in D$ is taken, and then $n \rightarrow \infty$.

Typical result $C(\mathcal{P}_{n,d}, D) \approx n^\sigma$.

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For **upper bounds** we will use **comparison** between domains, **affine transforms**, known results for **cube** and **ball**, and a new **idea**.

For **lower bounds** we will use the variational definition and will give **constructions of polynomials** (which begin with Chebyshev polynomials) that provide **matching bounds** in many situations.

Comparison between domains:

if $D_1 \subset D_2 \subset \mathbb{R}^d$ are two bounded domains, then

$$C(\mathcal{P}_{n,d}, D_1, \mathbf{x}) \geq C(\mathcal{P}_{n,d}, D_2, \mathbf{x}), \quad \mathbf{x} \in D_1.$$

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Affine transforms:

for any non-degenerate affine transform T ($T\mathbf{x} = \mathbf{x}_0 + A\mathbf{x}$, $\det T := \det A$) on \mathbb{R}^d and any bounded domain $D \subset \mathbb{R}^d$, we have

$$C(\mathcal{P}_{n,d}, T(D), T\mathbf{x}) = C(\mathcal{P}_{n,d}, D, \mathbf{x}) |\det T|^{-1}, \quad \mathbf{x} \in D.$$

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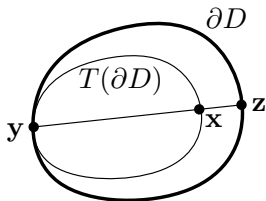
If $D \subset \mathbb{R}^d$ is a convex body, then $\max_{\mathbf{x} \in D} C(\mathcal{P}_n, D, \mathbf{x}) \leq 2^d \max_{\mathbf{x} \in \partial D} C(\mathcal{P}_n, D, \mathbf{x})$.

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Observation: if $\mathbf{x} \in D$, then for some affine transform T with $|\det T| \geq \frac{1}{2^d}$, we have

$$\mathbf{x} \in T(\partial D) \quad \text{and} \quad T(D) \subset D.$$

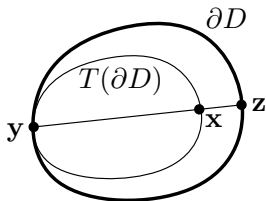


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Conjecture: if $D \subset \mathbb{R}^d$ is a convex body, then for each n , $\max_{\mathbf{x} \in D} C(\mathcal{P}_{n,d}, D, \mathbf{x})$ is attained at an extreme point of D .

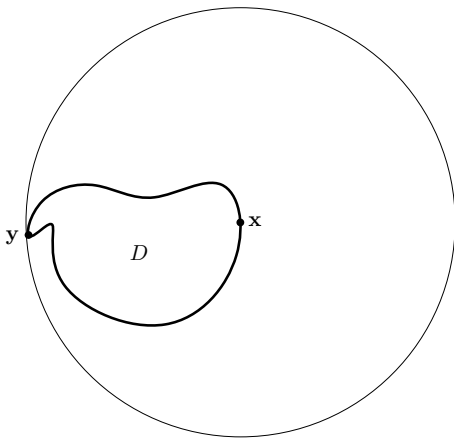
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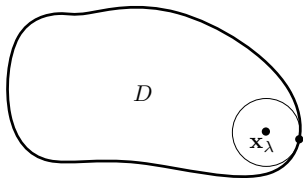
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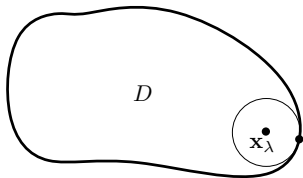
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Matching upper bound: if $D \subset \mathbb{R}^d$ is a compact set, and $D = \cup_{\lambda} (\mathbf{x}_{\lambda} + r_{\lambda} B_2^d)$ with $\inf_{\lambda} r_{\lambda} > 0$, then $C(\mathcal{P}_{n,d}, D) \leq cn^{d+1}$ (so $C(\mathcal{P}_{n,d}, D) \approx n^{d+1}$).

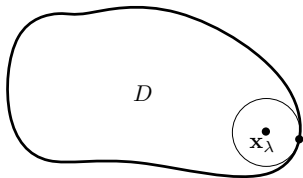


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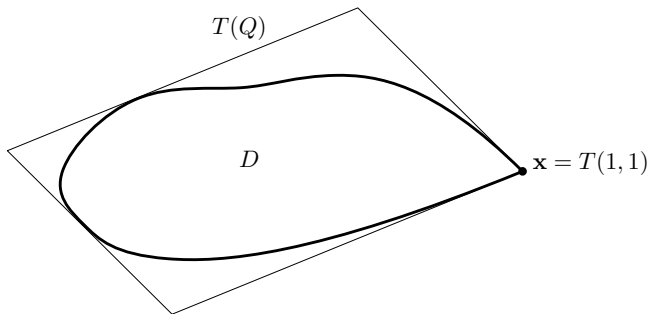
$$C(\mathcal{P}_{n,d}, B_{\alpha}^d) \approx n^{d+1} \text{ for } 2 \leq \alpha < \infty.$$

Cube: $Q = B_{\infty}^d = [-1, 1]^d$, $C(\mathcal{P}_{n,d}, Q) = C(\mathcal{P}_{n,d}, Q, (\pm 1, \dots, \pm 1)) \approx n^{2d}$.

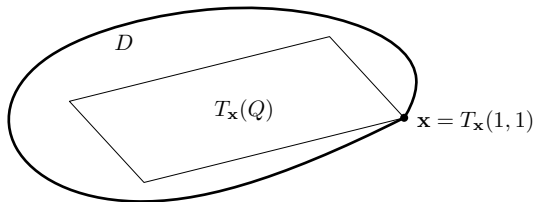
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Lower bound: if $D \subset \mathbb{R}^d$ is a compact set, T is an affine transform with $D \subset T(Q)$ and $\mathbf{x} := T(1, \dots, 1) \in D$, then

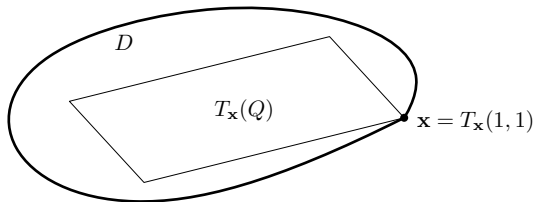
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Upper bound: if $D \subset \mathbb{R}^d$ is a compact set and for any $\mathbf{x} \in D$ there is $T_{\mathbf{x}}$ satisfying $\mathbf{x} = T_{\mathbf{x}}(1, \dots, 1)$, $T_{\mathbf{x}}(Q) \subset D$, while $\inf_{\mathbf{x}} |\det T_{\mathbf{x}}| > 0$, then $C(\mathcal{P}_{n,d}, D) \leq cn^{2d}$.

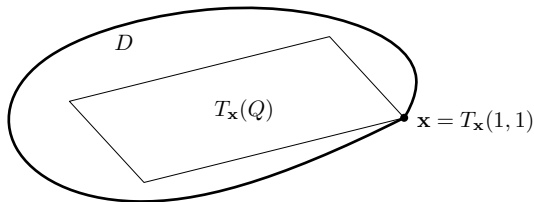


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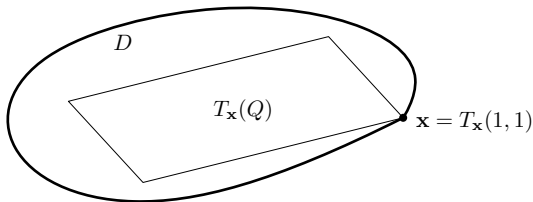
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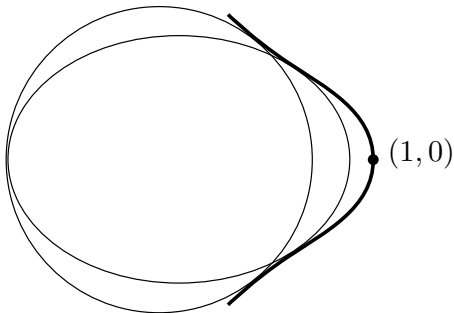
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Ellipsoid with “handle”

Define $\mathbf{v}_n^d := (1 + n^{-2}/3, 0, \dots, 0) \in \mathbb{R}^d$.

Recalling that $C(\mathcal{P}_{n,d}, B_2^d, (1, 0, \dots, 0)) \approx n^{d+1}$,

Markov's inequality implies $C(\mathcal{P}_{n,d}, B_2^d, \mathbf{v}_n^d) \leq c(d)n^{d+1}$.

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Theorem (main result for upper estimates). If $D \subset \mathbb{R}^d$ is a compact set, T is an affine transform of \mathbb{R}^d with $T(B_2^d) \subset D$, then

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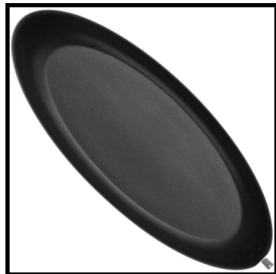
Recalling that $C(\mathcal{P}_{n,d}, B_2^d, (1, 0, \dots, 0)) \approx n^{d+1}$,

Markov's inequality implies $C(\mathcal{P}_{n,d}, B_2^d, \mathbf{v}_n^d) \leq c(d)n^{d+1}$.

Theorem (main result for upper estimates). If $D \subset \mathbb{R}^d$ is a compact set, T is an affine transform of \mathbb{R}^d with $T(B_2^d) \subset D$, then

$$C(\mathcal{P}_{n,d}, D, T\mathbf{v}_n^d) \leq c(d)|\det T|^{-1}n^{d+1}.$$

Example: $C(\mathcal{P}_{n,2}, [-1, 1]^2, (\pm 1, \pm 1)) \leq cn^4$.

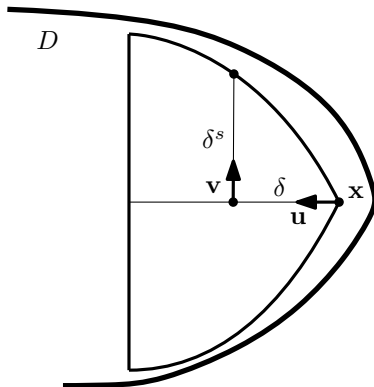


“Cone”-type sets

Theorem. If $D \subset \mathbb{R}^d$ is a compact set, $\mathbf{x} \in D$, $\mathbf{u} \in \mathbb{R}^d$, $|\mathbf{u}| = 1$, $\beta, \gamma > 0$, $s \in [\frac{1}{2}, 1]$, and

$$\{\mathbf{x} + \delta\mathbf{u} + \lambda\delta^s\mathbf{v} : \delta \in [0, \beta], \lambda \in [0, \gamma], \mathbf{v} \in \mathbb{R}^d, |\mathbf{v}| = 1, \mathbf{u} \perp \mathbf{v}\} \subset D,$$

then $C(\mathcal{P}_{n,d}, D, \mathbf{x}) \leq c(d, \beta, \gamma)n^{2+2s(d-1)}$.

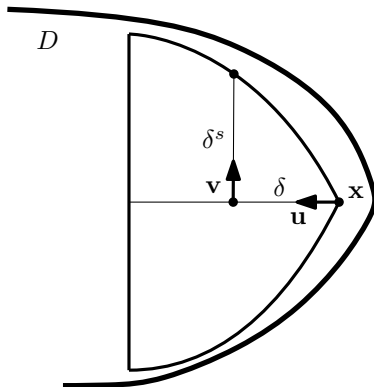


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$s = 1$: right circular cone, $s = 1/2$: almost a spherical cap.

Corollary: upper estimate on $C(\mathcal{P}_{n,d}, B_\alpha^d)$ for $1 < \alpha < 2$

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At any $\mathbf{x} \in \partial B_\alpha^d$ we can inscribe a “cone”-type set with $s = 1/\alpha$, “vertex” at \mathbf{x} , and “axis” in the direction opposite to the outward normal vector to the surface ∂B_α^d at \mathbf{x} .

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$$C(\mathcal{P}_{n,d}, B_\alpha^d) \leq c(\alpha, d)n^{2+\frac{2}{\alpha}(d-1)}, \quad 1 < \alpha < 2.$$

Lower estimates: basic univariate construction

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Denote $\rho_n(x) := \frac{1}{n^2} + \frac{1}{n}\sqrt{1-x^2}$. For any $n, m \geq 1$ and $y \in [-1, 1]$, there exists a polynomial $P_n = P_{n,m,y}$ of degree $\leq n$ such that $P_n(y) = 1$ and

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Two ways to obtain multivariate polynomials:
ridge functions and **tensor product**.

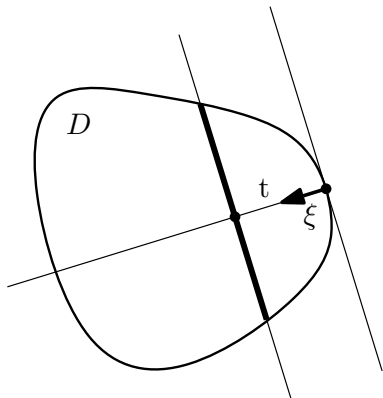
Lower estimate in terms of parallel section function

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For a unit vector $\xi \in \mathbb{R}^d$ and a non-empty compact set $D \subset \mathbb{R}^d$,

$$A_{D,\xi}(t) := \text{Vol}_{d-1}(D \cap \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \xi = t + h\}),$$

where $h \in \mathbb{R}$ is the smallest h such that $D \cap \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \xi = t + h\} \neq \emptyset$.



Theorem. If $D \subset \mathbb{R}^d$ is a compact set, $\xi \in \mathbb{R}^d$, $|\xi| = 1$, and $n, m \geq 1$, then

$$C(\mathcal{P}_n, D) \geq c(m, D) \left(\int_0^{n^{-2}} A_{D, \xi}(t) dt + n^{-2m} \int_{n^{-2}}^{\infty} A_{D, \xi}(t) t^{-m} dt \right)^{-1}.$$

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So, $C(\mathcal{P}_{n,d}, B_\alpha^d) \approx n^{2+\frac{2}{\alpha}(d-1)}$, $1 < \alpha < 2$.

Lower estimate using tensor product and affine transform

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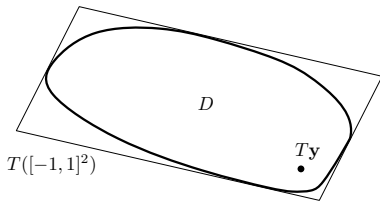
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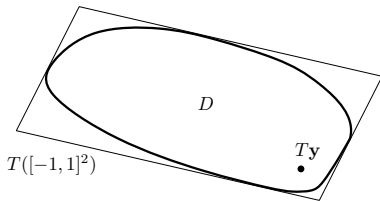
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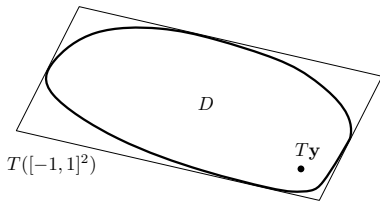


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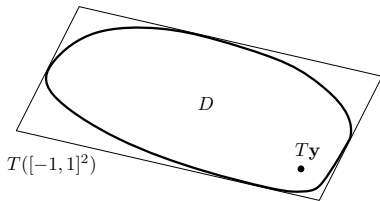


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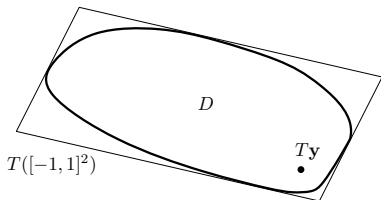


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Question: is it true that for any $\mathbf{x} \in D$

$$C(\mathcal{P}_{n,d}, D, \mathbf{x}) \leq c(D) \sup_T |\det T|^{-1} \rho_n^{-1}(y_1) \rho_n^{-1}(y_2) \cdots \rho_n^{-1}(y_d)$$

where the sup is taken over T satisfying $D \subset T([-1, 1]^d)$ and $T\mathbf{y} = \mathbf{x}$, for a large class of D , say for convex bodies?



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Half-balls in \mathbb{R}^d : $B_+^d := B_2^d \cap \{\mathbf{x} : x_1 \geq 0\}$, $C(\mathcal{P}_{n,d}, B_+^d) \approx n^{d+2}$.

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Affine transforms affect the constant, but not σ ,

say for a “thin” triangle A with vertices $(\pm 1, 0)$ and $(0, \varepsilon)$,
we get $C(\mathcal{P}_{n,2}, A) \approx \varepsilon^{-1} n^4$.

Thank you!

Reference:

- Z. Ditzian and A. Prymak, On Nikol'skii inequalities for domains in \mathbb{R}^d , <http://arxiv.org/abs/1409.5397>