Optimal polynomial meshes exist on any multivariate convex domain

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Optimal polynomial meshes

$\|f\|_E := \max_{\mathbf{x}\in E} |f(\mathbf{x})|$, where $E \subset \mathbb{R}^d$ is a compact set and $f \in C(E)$.
$\mathcal{P}_n^d$ — real algebraic polynomials in d variables of total degree at most n.
Note $\dim \mathcal{P}_n^d \approx n^d$.

A compact $\Omega \subset \mathbb{R}^d$ possesses *optimal polynomial meshes* if
$\exists \{Y_n\}_{n\geq 1}$, $Y_n \subset \Omega$, $\# Y_n \leq C_1 n^d$ such that

$$\|P\|_\Omega \leq C_2 \|P\|_{Y_n} \quad \forall P \in \mathcal{P}_n^d, \tag{1}$$

where C_1, C_2 depend only on Ω.

Applications in: discrete least squares approximation, cubature formulas, scattered data interpolation, study of discrete Fekete and Leja type sets.

A constructive proof of existence usually involves some polynomial inequalities which are of independent interest.

Example: segment

$$|P'(x)| \le \min\left\{n^2, \frac{n}{\sqrt{1-x^2}}\right\}\|P\|_{[-1,1]} \quad \forall P \in \mathcal{P}_n^1$$

For Chebyshev partition $\{\cos\frac{k\pi}{N}\}_{k=0}^{N}$, $N = \ell n$, this Bernstein-Markov factor behaves as $\frac{o(\ell)}{|I|}$ when $x \in I$, $\ell \to \infty$, where I is any interval of the partition.

$$\max_{x \in I} |P'(x)||I| \le \frac{1}{2}\|P\|_{[-1,1]}$$

From each I select a point to include into Y_n. Then for any $P \in \mathcal{P}_n^1$ we can find z with $|P(z)| = \|P\|_{[-1,1]}$ so if $z \in I$ and $y \in I \cap Y_n$, by MVT

$$\|P\|_{Y_n} \ge |P(y)| \ge \|P\|_{[-1,1]} - \max_{x \in I}|P'(x)||I| \ge \frac{1}{2}\|P\|_{[-1,1]}.$$

History of the problem

2012 Bloom, Bos, Calvi, Levenberg: *nearly* optimal meshes, i.e. those satisfying (1) with $\#Y_n \leq C(n \log n)^d$, exist for any compact $\Omega \subset \mathbb{R}^d$. (Fekete points used, essentially non-constructive proof.)

Existence (constructive) of optimal polynomial meshes
2011, 2013 Kroo: convex polytopes, C^α star-like domains with $\alpha > 2 - \frac{2}{d}$.
Conjecture: any convex body in $\mathbb{R}^d$***,*** $d \geq 2$***.***
2016 Piazzon: certain extension of C^2 domains.
2019 Kroo: any convex body in $\mathbb{R}^2$ using a new tangential Bernstein type inequality.
2021 P.: alternative proof for any convex body in $\mathbb{R}^2$ via a connection between the Christoffel functions, positive quadrature formulas and polynomial meshes established in the paper [2] of Bos and Vianello (2019).

Main result

We confirm Kroo's conjecture for $d \geq 3$.

Theorem

For any convex body $\Omega \subset \mathbb{R}^d$*, there exists* $\{Y_n\}_{n \geq 1}$*,* $Y_n \subset \Omega$*,* $\# Y_n \leq C(d) n^d$ *such that*

$$\|P\|_\Omega \leq 2\|P\|_{Y_n} \quad \forall P \in \mathcal{P}_n^d. \tag{2}$$

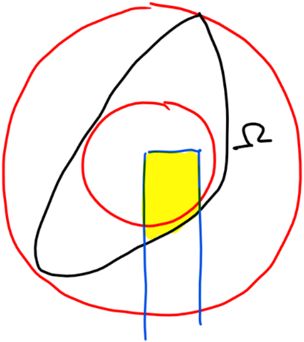
ε-version is also valid: 2 can be replaced with $1+\varepsilon$ and $C(d)$ with $C(d,\varepsilon)$.

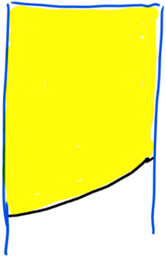
John's theorem, graph-type domains

Optimal meshes are invariant under affine transforms, so John's theorem on inscribed ellipsoid of the largest volume allows to assume

$$B(\mathbf{0}, 1) \subset \Omega \subset B(\mathbf{0}, d) \tag{3}$$

and represent Ω as the union of graph-type domains with good parameters.



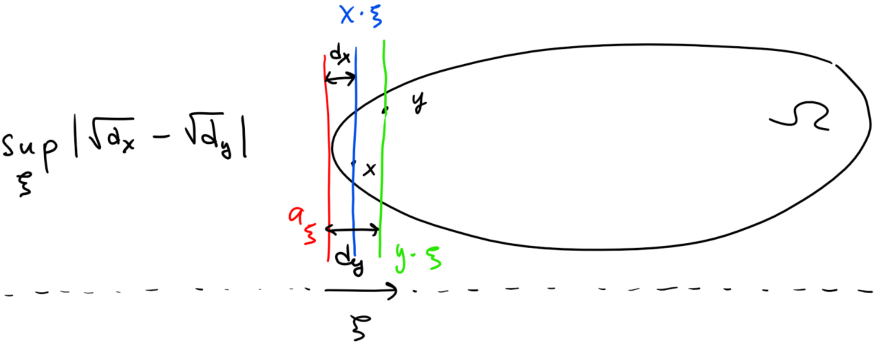


Dubiner's metric

Let $\Omega \subset \mathbb{R}^d$ be a convex body. For any $\mathbf{x}, \mathbf{y} \in \Omega$, define

$$\rho(\mathbf{x}, \mathbf{y}) := \rho_\Omega(\mathbf{x}, \mathbf{y}) := \max_{\boldsymbol{\xi} \in \mathbb{S}^{d-1}} \left| \sqrt{\mathbf{x} \cdot \boldsymbol{\xi} - a_{\boldsymbol{\xi}}} - \sqrt{\mathbf{y} \cdot \boldsymbol{\xi} - a_{\boldsymbol{\xi}}} \right|, \qquad (4)$$

where $a_{\boldsymbol{\xi}} := \min_{\mathbf{z} \in \Omega} \mathbf{z} \cdot \boldsymbol{\xi}$, and $\mathbb{S}^{d-1}$ denotes the unit sphere of $\mathbb{R}^d$.

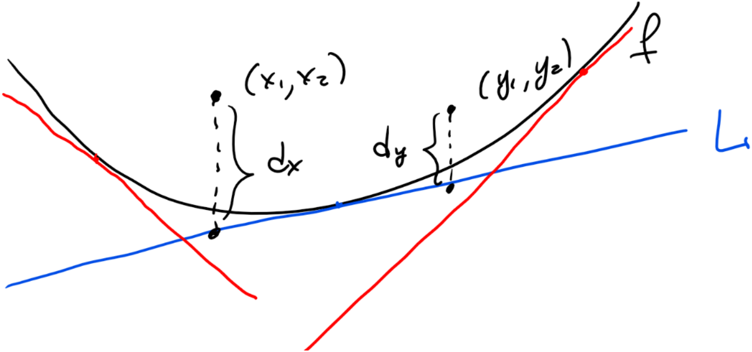


Equivalent metric for special subdomains

When $d = 2$ for a special subdomain G which essentially the epigraph of f an equivalent metric will be

$$\rho_G((x_1, x_2), (y_1, y_2)) = \max_L \left| \sqrt{x_2 - L(x_1)} - \sqrt{y_2 - L(y_1)} \right|$$

with maximum over all supporting lines L to f.



Construction of the mesh

There exists a small $c_d > 0$ such that for arbitrary $0 < \eta \leq c_d$ one can take Λ_n as any η/n-separated set which is c_d/n-covering w.r.t. ρ, i.e.:
(separation) $\forall x, y \in \Lambda_n$, $x \neq y$: $\rho(x, y) \geq \eta/n$;
(covering) $\forall z \in \Omega \; \exists x \in \Lambda_n$: $\rho(z, x) \leq c_d/n$.

When $\eta = c_d$, the above conditions mean that Λ_n is a maximal c_d/n-separated set in Ω w.r.t. ρ. Choosing $\eta < c_d$ allows more flexibility in construction of Λ_n.

Variation of polynomials

We obtain the estimate (2) as an immediate consequence of the covering condition and the following:

Theorem

Let $\Omega \subset \mathbb{R}^d$ be a convex body satisfying (3). There exists $C_*(d) > 0$ such that for any $Q \in \mathcal{P}_n^d$ with $\|Q\|_\Omega \leq 1$, we have

$$|Q(\mathbf{x}) - Q(\mathbf{y})| \leq C_*(d) n \rho(\mathbf{x}, \mathbf{y}) \quad \textit{whenever} \quad \mathbf{x}, \mathbf{y} \in \Omega. \tag{5}$$

The proof of Theorem 2 is essentially two-dimensional which is due to the fact that any supporting line to a 2-dimensional section of Ω can be extended to a supporting hyperplane for Ω.

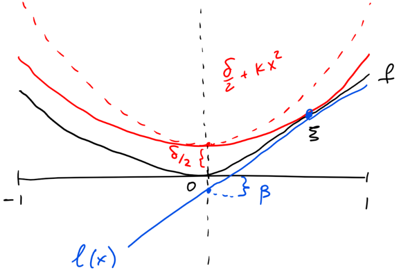
Key planar lemma

Lemma

If $f : [-1,1] \to [0,\infty)$ *is convex with* $f(0) = 0$ *and* $\max_{x\in[-1,1]} f(x) > \frac{\delta}{2}$, *then there exist* $k > 0$, $\xi \in [-1,1] \setminus \{0\}$, *and* $\ell(x) := \alpha x - \beta$ *satisfying:*

$$0 < |\alpha| \leq \frac{1}{3},\ 0 \leq \beta \leq \frac{1}{3},\ \ell(\xi) = f(\xi),\ \ell'(\xi) = f'_{-}(\xi),$$

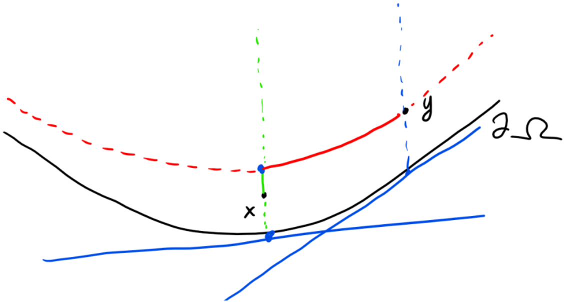
$$f(x) \leq \frac{\delta}{2} + kx^2 \ \ \forall x \in [-1,1], \quad \frac{\sqrt{\delta+\beta}}{|\alpha|} < \frac{1}{\sqrt{k}}.$$



Key planar lemma: consequences

We find that it suffices to use $\rho(\mathbf{x}, \mathbf{y})$ with the maximum in (4) being taken over only two specific directions which are inward normal vectors at two boundary points of Ω. ($x = 0$ and $x = \xi$)

This allows us to explicitly construct a parallelogram in G containing both $\mathbf{x}$ and $\mathbf{y}$, as well as certain families of parabolas and straight segments in G along which the standard one-dimensional Bernstein inequality can be applied to derive (5).



Multivariate fast decreasing polynomials

To complete the proof, it suffices to show $\#Y_n \leq \dim \mathcal{P}^d_{C^* n}$ for some large $C^* = C^*(d)$. Indeed, if this were not true, then one can get a contradiction using linear dependence of the "fast decreasing" polynomials provided by the following theorem for each $\mathbf{x} \in Y_n$.

Theorem

Let $\Omega \subset \mathbb{R}^d$ be a convex body satisfying (3). For any $\mathbf{x} \in \Omega$ and $n \in \mathbb{N}$, there exists a polynomial $P \in \mathcal{P}^d_n$ such that $P(\mathbf{x}) = 1$ and

$$0 \leq P(\mathbf{z}) \leq C \exp(-c\sqrt{n\rho(\mathbf{x},\mathbf{z})}) \quad \text{for any} \quad \mathbf{z} \in \Omega, \tag{6}$$

where $C > 1$ and $c \in (0,1)$ are constants depending only on d.

Doubling property

An important ingredient for the proof of this theorem and for obtaining the above mentioned contradiction is the doubling property of ρ.

Theorem

Let $\Omega \subset \mathbb{R}^d$ be a convex body satisfying (3). For any $\mathbf{x} \in \Omega$ and $h > 0$

$$|B_\rho(\mathbf{x}, 2h)| \leq 4^d |B_\rho(\mathbf{x}, h)|. \tag{7}$$

The proof is essentially one-dimensional by considering strips

$$S(\Omega, \mathbf{x}, \boldsymbol{\xi}, h) := \left\{ \mathbf{y} \in \Omega : \quad \left| \sqrt{\mathbf{x} \cdot \boldsymbol{\xi} - a_{\boldsymbol{\xi}}} - \sqrt{\mathbf{y} \cdot \boldsymbol{\xi} - a_{\boldsymbol{\xi}}} \right| \leq h \right\}$$

and showing

$$S(\Omega, \mathbf{x}, \boldsymbol{\xi}, 2h) \subset \varphi\Big(S(\Omega, \mathbf{x}, \boldsymbol{\xi}, h)\Big), \quad \mathbf{x} \in \Omega, \quad \boldsymbol{\xi} \in \mathbb{S}^{d-1},$$

where $\varphi(\mathbf{z}) := \mathbf{x} + 4(\mathbf{z} - \mathbf{x})$.

References

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Thank you!