Geometry of Banach spaces and sharp versions of Jackson and Marchaud inequalities

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joint work with Zeev Ditzian

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Classical Jackson estimate for approximation by trigonometric polynomials on $[-\pi, \pi] = T$

$$E_n(f)_p \le c\omega^r(f, n^{-1})_p$$

was sharpened by **M. F. Timan (1966):**

$$n^{-r}\Big\{\sum_{k=1}^{n} k^{sr-1} E_k(f)_p^s\Big\}^{\frac{1}{s}} \le c\omega^r(f, n^{-1})_p, \quad 1 < p < \infty,\ s = \max(p, 2).$$

Here $\omega^r(f,t)_p = \sup_{|h| \le t} \|\Delta_h^r f\|_{L_p(T)}$,

$\Delta_h f(x) = f(x+h) - f(x),\ \Delta_h^r f(x) = \Delta_h \Delta_h^{r-1} f(x)$ and

$E_k(f)_p = \min_{\deg T_n < k} \|f - T_n\|_{L_p(T)}$.

F. Dai, Z. Ditzian, S. Tikhonov (2008) proved a general result for sharp Jackson estimates using a version of the *Littlewood-Paley inequality*. Examples of application include approximation by:

• algebraic polynomials on $[-1, 1]$,
• spherical harmonic polynomials on S^{d-1},
• functions of exponential type on $\mathbb{R}^d$,
• multivariate trigonometric polynomials on T^d.

For $1 < p < \infty$, $s = \max(p, 2)$, the sharp Jackson inequality

$$n^{-r}\left\{\sum_{k=1}^{n} k^{sr-1} E_k(f)_p^s\right\}^{\frac{1}{s}} \le c\omega^r(f, n^{-1})_p$$

is essentially equivalent to the following sharp lower estimate of $\omega^r(f, t)_p$ by $\omega^{r+1}(f, u)_p$

$$t^r\left\{\int_t^1 u^{-sr-1}\omega^{r+1}(f, u)_p^s\, du\right\}^{\frac{1}{s}} \le c\omega^r(f, t)_p.$$

The well-known immediate (but much weaker) lower bound is

$$\omega^{r+1}(f, t)_p \le 2\omega^r(f, t)_p.$$

In the other direction, for $1 < p < \infty$, with $q = \min(p, 2)$, the sharp converse inequality

$$\omega^r(f, n^{-1})_p \le cn^{-r}\left\{\sum_{k=1}^{n} k^{qr-1} E_k(f)_p^q\right\}^{\frac{1}{q}}$$

is essentially equivalent to the sharp Marchaud inequality

$$\omega^r(f, t)_p \le ct^r\left\{\int_t^1 u^{-qr-1}\omega^{r+1}(f, u)_p^q \, du\right\}^{\frac{1}{q}}.$$

Note that if $p = 2$, then $q = s = 2$, and we obtain an equivalence.

$$\omega^r(f,t)_2 \approx t^r \Big\{ \int_t^1 u^{-2r-1} \omega^{r+1}(f,u)_2^2 \, du \Big\}^{\frac{1}{2}}.$$

Z. Ditzian (1988) proved the sharp Marchaud inequality for Banach spaces B of functions on $\mathbb{R}^d$ or T^d for which translations are continuous isometries and for some $1 < q \le 2$ and $K > 1$

$$\frac{\|f+g\|_B + \|f-g\|_B}{2} \le (\|f\|_B^q + K\,\|g\|_B^q)^{\frac{1}{q}}, \forall f, g \in B,$$

and showed this condition to be equivalent to

$$\sup_{\substack{\|f\|_B=1 \\ \|g\|_B=t}} \left(\frac{\|f+g\|_B + \|f-g\|_B}{2} - 1 \right) \le ct^q, \; t > 0,$$

which means that B has modulus of smoothness of power type q in terminology of geometry of Banach spaces.

For L_p spaces, $1 < p < \infty$, we have $q = \min\{p, 2\}$.

Joint work with Z. Ditzian (2007): sharp Marchaud and converse inequalities in Orlicz spaces for which $\Phi(u^{\frac{1}{q}})$ is convex for some q, $1 < q \leq 2$, where $\Phi(u)$ is the Orlicz function.
The condition

$$\frac{\|f+g\|_B + \|f-g\|_B}{2} \leq (\|f\|_B^q + K\|g\|_B^q)^{\frac{1}{q}}, \forall f, g \in B,$$

was obtained for an *equivalent* norm.

Joint work with Z. Ditzian (2011): sharp Jackson and lower estimates of $\omega^r(f,t)_B$ are achieved for Banach spaces B of functions on $\mathbb{R}^d$ or T^d (or S^{d-1}) for which translations (rotations) are continuous isometries and for some s, $2 \le s < \infty$ and $k > 0$

$$\max(\|f+g\|_B, \|f-g\|_B) \ge (\|f\|_B^s + k\|g\|_B^s)^{\frac{1}{s}}, \forall f, g \in B.$$

For $q^{-1}+s^{-1}=1$, the dual space B^* satisfies

$$\frac{\|f+g\|_{B^*}+\|f-g\|_{B^*}}{2} \le (\|f\|_{B^*}^q + K\|g\|_{B^*}^q)^{\frac{1}{q}}, \forall f,g \in B^*,$$

if and only if

$$\max(\|f+g\|_B, \|f-g\|_B) \ge (\|f\|_B^s + k\|g\|_B^s)^{\frac{1}{s}}, \forall f,g \in B.$$

This establishes that the last condition is equivalent to

$$\inf_{\substack{\|f\|_B=\|g\|_B=1 \\ \|f-g\|_B=\varepsilon}} \left(1-\frac{\|f+g\|_B}{2}\right) \ge c\varepsilon^s,\ \varepsilon>0,$$

which means that B has modulus of convexity of power type s in terminology of geometry of Banach spaces.

In summary, certain geometric property of an *equivalent* norm of a Banach space implies an approximation inequality in the space.

Modulus of smoothness of power type q, $1 < q \le 2$, implies the sharp Marchaud inequality (upper estimate of ω^r in terms of ω^{r+1}).

Modulus of convexity of power type s, $2 \le s < \infty$, implies the sharp Jackson inequality (lower estimate of ω^r in terms of ω^{r+1}).

Joint work with Z. Ditzian (2011): sharp Jackson and lower estimates of $\omega^r(f,t)_{O(\Phi)}$ are achieved for Orlicz spaces for which $\Phi(u^{\frac{1}{s}})$ is concave for some s, $2 \le s < \infty$, where $\Phi(u)$ is the Orlicz function. In fact, it is sufficient to require that $\Phi(u^{\frac{1}{s}})$ is concave on $[0,a]$ and on $[b,\infty)$ for some $0 < a < b$.

Examples:
- $\Phi(u) = \max\{u^\alpha, u^\beta\}$, $1 < \alpha < \beta$, $s \ge \max\{2,\beta\}$;
- $\Phi(u) = u^r(1 + |\ln u|)$, $r \ge (3+\sqrt{5})/2$, $s > r$;
- Zygmund spaces $L_p(\mathrm{Log}L)^\alpha$:

$\Phi(u) = u^p(\ln(2+u))^{\alpha p}$, $\alpha p \ge 1$, $p \ge 1$, $s > p$, $s \ge 2$.

Theorem.

Suppose B is a Banach space of functions on $\mathbb{R}_+$, $\mathbb{R}$ or T satisfying

$$\max(\|f+g\|_B, \|f-g\|_B) \geq (\|f\|_B^s + k\|g\|_B^s)^{\frac{1}{s}}, \forall f, g \in B$$

for some $2 \leq s < \infty$ and $\|f(\cdot+\xi)\|_B \leq \|f(\cdot)\|_B$ for $\xi > 0$. Then

$$\left\{\sum_{j=1}^{\infty} 2^{-jrs} \omega^{r+1}(f, 2^j t)_B^s\right\}^{\frac{1}{s}} \leq c\omega^r(f,t)_B.$$

Our results also cover:

• C_0 semigroups of contraction operators,

• sharp Jackson estimates for approximation by algebraic polynomials on a simplex in $\mathbb{R}^d$,

• sharp Jackson inequality on the sphere S^{d-1}.

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