Non-existence of \((76, 30, 8, 14)\) strongly regular graph and some structural tools

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joint work with Andrii Bondarenko and Danylo Radchenko
A finite, undirected, simple, \( k \)-regular graph \( G = (V, E) \) on \( v \) vertices is strongly regular with parameters \((v, k, \lambda, \mu)\) if \( |N(i) \cap N(j)| = \lambda \) for any two adjacent \( i, j \in V \) and \( |N(i) \cap N(j)| = \mu \) for any two non-adjacent \( i, j \in V \).
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A finite, undirected, simple, $k$-regular graph $G = (V, E)$ on $v$ vertices is strongly regular with parameters $(v, k, \lambda, \mu)$ if $|N(i) \cap N(j)| = \lambda$ for any two adjacent $i, j \in V$ and $|N(i) \cap N(j)| = \mu$ for any two non-adjacent $i, j \in V$. For example, Petersen graph is a $(10, 3, 0, 1)$ SRG.

Strong regularity can be expressed in terms of the incidence matrix $A$ as

$$AJ = kJ, \quad A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J,$$

leading to $(v - k - 1)\mu = k(k - \lambda - 1)$ and the eigenvalues

$$k \quad \text{of multiplicity 1,}$$

$$r = \frac{1}{2} \left( \lambda - \mu + \sqrt{ (\lambda - \mu)^2 + 4(k - \mu) } \right) \quad \text{of multiplicity } f = \frac{1}{2} \left( v - 1 - \frac{2k + (v - 1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}} \right),$$

$$s = \frac{1}{2} \left( \lambda - \mu - \sqrt{ (\lambda - \mu)^2 + 4(k - \mu) } \right) \quad \text{of multiplicity } g = \frac{1}{2} \left( v - 1 + \frac{2k + (v - 1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}} \right).$$
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This defines a suitable family of parameters $(v, k, \lambda, \mu)$ for which the corresponding SRG may exist.
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Unknown cases for small $v$:

(65, 32, 15, 16),
(69, 20, 7, 5),
(75, 32, 10, 16),
(76, 30, 8, 14) — our case,
(76, 35, 18, 14),
ten more cases with $85 \leq v \leq 100$. 

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Note that the complement of a $(v, k, \lambda, \mu)$ SRG is a $(v, v - 1 - k, v - 2k + \mu - 2, v - 2k + \lambda)$ SRG.
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**Theorem.** There is no $(76, 30, 8, 14)$ SRG.
Outline of the proof:

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- $K_4$ is a subgraph of $(76, 30, 8, 14)$ SRG;
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Techniques:

- Euclidean representation of SRGs by systems of unit vectors;
- positive semidefiniteness (and rank) of Gram matrix;
- Cauchy-Shwartz inequality in the space of spherical harmonics;
- projections of Euclidean representation;
- various combinatorial counting arguments;
- two insignificant computer searches.
Euclidean representation \( \{x_i : i \in V\} \) of a SRG \( G \) in \( \mathbb{R}^g \)

\[ A - sl \geq 0, \text{ therefore } A - sl = (z_i \cdot z_j)_{i,j \in V} \text{ for some } \{z_i : i \in V\} \subset \mathbb{R}^g; \]

define \( x_i := z_i / \| z_i \|, i \in V \).
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\[
    x_i \cdot x_j = \begin{cases} 
    1, & \text{if } i = j, \\
    p, & \text{if } i \text{ and } j \text{ are adjacent,} \\
    q, & \text{otherwise,}
    \end{cases}
\]

where \( p = s/k \), and \( q = -(s+1)/(v-k-1) \).
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For \((v, k, \lambda, \mu) = (76, 30, 8, 14)\), we have \( r^f = 2^{57} \) and \( s^g = (-8)^{18} \).

\( \ln \mathbb{R}^{18}, (p, q) = (-\frac{4}{15}, \frac{7}{45}) \).

\( \ln \mathbb{R}^{57}, (p, q) = (\frac{1}{15}, -\frac{1}{15}) \) (complement).
Positive semidefiniteness of Gram matrix

Let $\pi = \{ V_j \}_{j=1}^\ell$ be a partition of a subset $\tilde{V} \subset V$. Set $X_j := \sum_{t \in V_j} x_t$, $j = 1, \ldots, \ell$, and let $M = (X_i \cdot X_j)_{i,j=1}^\ell$. 

We extensively used the following important inequality: $\det M \geq 0$.

Example: there is no $K_5$ in $(76, 30, 8, 14)$ SRG.

Recall that in $\mathbb{R}^{18}$, we have $(p, q) = (-4, 7)$. If $V_1$ are the vertices of $K_5$, with $\ell = 1$ we get a contradiction because $\det M = M_{1,1} = 5 + 20p = -1/3$. 

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If $a_{i,j}$ is the number of edges between $V_i$ and $V_j$, $i \neq j$, and $a_{i,i}$ is the number of edges in the subgraph induced by $V_i$, then

$$M_{i,i} = |V_i| + 2a_{i,i}p + (|V_i||V_i| - 1) - 2a_{i,i}q,$$
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Lower bound on the number of 4-cliques in a SRG: Preliminaries from harmonic analysis
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For fixed integers $n \geq 1$, $t \geq 0$, consider homogeneous polynomials

$$P(x_1, \ldots, x_n) = \sum_{t_1, \ldots, t_n \geq 0 \atop t_1 + \cdots + t_n = t} \alpha_{t_1, \ldots, t_n} x_1^{t_1} \cdots x_n^{t_n}$$

satisfying the Laplace’s equation

$$\Delta P = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} P = 0.$$ 

**Spherical harmonic** of degree $t$ is the restriction of a polynomial satisfying the above properties to the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. 

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Let $\mathcal{P}_{n,t}$ be the inner product space of all spherical harmonics of degree $t$ on $S^{n-1}$ with

$$\langle P, Q \rangle = \int_{S^{n-1}} P(z) Q(z) d\mu_n(z).$$
Riesz representation theorem gives a natural mapping \( S^{n-1} \ni x \mapsto P_x \in \mathcal{P}_{n,t} \) such that

\[
\langle P_x, Q \rangle = \int_{S^{n-1}} P_x(z)Q(z) \, d\mu_n(z) = Q(x) \quad \text{for all} \quad Q \in \mathcal{P}_{n,t}.
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It is well-known that
\[
\langle P_x, P_y \rangle = Z_{n,t}(x \cdot y), \quad x, y \in S^{n-1},
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where $Z_{n,t}(\xi) = \frac{2t+n-2}{n-2} C^{(n-2)/2}_t(\xi)$ is a zonal harmonic and $C^{(\alpha)}_t(\xi)$ are the Gegenbauer polynomials.
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\( \{ C_t^{(\alpha)}(\xi) \}_{t \geq 0} \) are orthogonal on \([-1, 1] \) with the weight \((1 - \xi^2)^{\alpha-1/2}\), and

\[
\frac{1 - z^2}{(1 - 2\xi z + z^2)^{\alpha+1}} = \sum_{t=0}^{\infty} \frac{t + \alpha}{\alpha} C_t^{(\alpha)}(\xi)z^t.
\]
For any finite sets of points \( \{x_i\}_{i \in I} \) and \( \{y_j\}_{j \in J} \) from \( S^{n-1} \), using the Cauchy-Shwartz inequality in \( P_{n,t} \), we obtain

\[
\left( \sum_{i \in I} \sum_{j \in J} \langle P_{x_i}, P_{y_j} \rangle \right)^2 = \left\langle \sum_{i \in I} P_{x_i}, \sum_{j \in J} P_{y_j} \right\rangle^2 \\
\leq \left\langle \sum_{i \in I} P_{x_i}, \sum_{i \in I} P_{x_i} \right\rangle \left\langle \sum_{j \in J} P_{y_j}, \sum_{j \in J} P_{y_j} \right\rangle \\
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$$\leq \left( \sum_{i \in I} P_{x_i}, \sum_{i \in I} P_{x_i} \right) \left( \sum_{j \in J} P_{y_j}, \sum_{j \in J} P_{y_j} \right)$$

$$= \sum_{i \in I} \sum_{i' \in I} \langle P_{x_i}, P_{x_i'} \rangle \sum_{j \in J} \sum_{j' \in J} \langle P_{y_j}, P_{y_{j'}} \rangle.$$ 

Rewriting this in terms of $Z_{n,t}$ gives the inequality

$$\left( \sum_{i \in I, j \in J} Z_{n,t}(x_i \cdot y_j) \right)^2 \leq \left( \sum_{i, i' \in I} Z_{n,t}(x_i \cdot x_{i'}) \right) \left( \sum_{j, j' \in J} Z_{n,t}(y_j \cdot y_{j'}) \right). \quad (*)$$
We take \( n := g \), 
\( \mathcal{I} := V, \ x_i \in S^{n-1} \subset \mathbb{R}^g \) be the Euclidean representation of \( i \), and 
\( \mathcal{J} := E, \ y_j := \frac{x_{j(1)}+x_{j(2)}}{\|x_{j(1)}+x_{j(2)}\|} \in S^{n-1}, \) where edge \( j \) joins \( j^{(1)}, j^{(2)} \in V \).
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For our \((76, 30, 8, 14)\) SRG, with \( t = 4 \), \( Z_{18,4}(\xi) = 54 - 2160\xi^2 + 7920\xi^4 \), we obtain \( N \geq \frac{2128}{55} > 38 \).
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For $t = 4$, the resulting bound on $N$ can be expressed in terms of a rational function of $k$, $r$, $s$ of degree $\leq 10$ in each variable.
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It is possible to prove non-existence of \((460, 153, 32, 60)\) SRG using our bound (work in preparation).
We take \( n := g \), 
\( \mathcal{I} := V \), \( x_i \in S^{n-1} \subset \mathbb{R}^g \) be the Euclidean representation of \( i \), and 
\( \mathcal{J} := E \), \( y_j := \frac{x_{j(1)} + x_{j(2)}}{\|x_{j(1)} + x_{j(2)}\|} \in S^{n-1} \), where edge \( j \) joins \( j^{(1)}, j^{(2)} \in V \).

If \( N \) is the number of 4-cliques in \( G \), then using strong regularity of \( G \), the inequality (*) reduces to a linear inequality on \( N \).

For our \((76, 30, 8, 14)\) SRG, with \( t = 4 \), \( Z_{18,4}(\xi) = 54 - 2160\xi^2 + 7920\xi^4 \), we obtain \( N \geq \frac{2128}{55} > 38 \).

For \( t = 4 \), the resulting bound on \( N \) can be expressed in terms of a rational function of \( k, r, s \) of degree \( \leq 10 \) in each variable.

It is possible to prove non-existence of \((460, 153, 32, 60)\) SRG using our bound (work in preparation).

Positive linear combinations of Gegenbauer polynomials are essentially our only choices of zonal functions (Schoenberg).
Outline of the proof:

- $K_4$ is a subgraph of $(76, 30, 8, 14)$ SRG;
- $(76, 30, 8, 14)$ SRG contains (as an induced subgraph) one of the three “larger” subgraphs;
- eliminating the case of $(40, 12, 2, 4)$ SRG;
- eliminating the case of $\overline{K_{16}}$;
- eliminating the case of $K_{6,10}$.

Techniques:

- Euclidean representation of SRGs by systems of unit vectors;
- positive semidefiniteness (and rank) of Gram matrix;
- Cauchy-Shwartz inequality in the space of spherical harmonics;
- projections of Euclidean representation;
- various combinatorial counting arguments;
- two insignificant computer searches.
References:

- A. V. Bondarenko, A. Prymak, and D. Radchenko, Supplementary files for the proof of non-existence of $SRG(76, 30, 8, 14)$, http://prymak.net/SRG-76-30-8-14/