Non-existence of $(76, 30, 8, 14)$ strongly regular graph and some structural tools

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joint work with Andrii Bondarenko and Danylo Radchenko

A finite, undirected, simple, k-regular graph $G = (V, E)$ on v vertices is strongly regular with parameters (v, k, λ, μ) if $|N(i) \cap N(j)| = \lambda$ for any two adjacent $i, j \in V$ and $|N(i) \cap N(j)| = \mu$ for any two non-adjacent $i, j \in V$.

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Strong regularity can be expressed in terms of the incidence matrix A as

$$AJ = kJ, \quad A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J,$$

leading to $(v - k - 1)\mu = k(k - \lambda - 1)$ and the eigenvalues

$$\begin{aligned} & k && \text{of multiplicity } 1, \\ r &= \frac{1}{2}\left(\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right) && \text{of multiplicity } f = \frac{1}{2}\left(v - 1 - \frac{2k + (v-1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}\right), \\ s &= \frac{1}{2}\left(\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right) && \text{of multiplicity } g = \frac{1}{2}\left(v - 1 + \frac{2k + (v-1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}\right). \end{aligned}$$

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This defines a suitable family of parameters (v, k, λ, μ) for which the corresponding SRG may exist.

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Unknown cases for small v:
$(65, 32, 15, 16)$,
$(69, 20, 7, 5)$,
$(75, 32, 10, 16)$,
$(76, 30, 8, 14)$ — our case,
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Theorem. There is no $(76, 30, 8, 14)$ SRG.

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Techniques:

- Euclidean representation of SRGs by systems of unit vectors;
- positive semidefiniteness (and rank) of Gram matrix;
- Cauchy-Shwartz inequality in the space of spherical harmonics;
- projections of Euclidean representation;
- various combinatorial counting arguments;
- two insignificant computer searches.

Euclidean representation $\{x_i : i \in V\}$ of a SRG G in $\mathbb{R}^g$

$A - sI \geq 0$, therefore $A - sI = (z_i \cdot z_j)_{i,j \in V}$ for some $\{z_i : i \in V\} \subset \mathbb{R}^g$; define $x_i := z_i/\|z_i\|$, $i \in V$.

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$$x_i \cdot x_j = \begin{cases} 1, & \text{if } i = j, \\ p, & \text{if } i \text{ and } j \text{ are adjacent}, \\ q, & \text{otherwise}, \end{cases}$$

where $p = s/k$, and $q = -(s+1)/(v-k-1)$.

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For $(v, k, \lambda, \mu) = (76, 30, 8, 14)$, we have $r^f = 2^{57}$ and $s^g = (-8)^{18}$.
In $\mathbb{R}^{18}$, $(p, q) = (-\frac{4}{15}, \frac{7}{45})$.
In $\mathbb{R}^{57}$, $(p, q) = (\frac{1}{15}, -\frac{1}{15})$ (complement).

Positive semidefiniteness of Gram matrix

Let $\pi = \{V_j\}_{j=1}^{\ell}$ be a partition of a subset $\widetilde{V} \subset V$.
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and $a_{i,i}$ is the number of edges in the subgraph induced by V_i, then

$$M_{i,i} = |V_i| + 2a_{i,i}p + (|V_i|(|V_i| - 1) - 2a_{i,i})q, \quad M_{i,j} = a_{i,j}p + (|V_i||V_j| - a_{i,j})q.$$

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Recall that in $\mathbb{R}^{18}$, we have $(p, q) = (-\frac{4}{15}, \frac{7}{45})$.
If V_1 are the vertices of K_5, with $\ell = 1$ we get a contradiction because

$$\det M = M_{1,1} = 5 + 20p = -1/3.$$

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For fixed integers $n \geq 1$, $t \geq 0$, consider homogeneous polynomials

$$P(x_1, \dots, x_n) = \sum_{\substack{t_1, \dots, t_n \geq 0 \\ t_1 + \dots + t_n = t}} \alpha_{t_1, \dots, t_n} x_1^{t_1} \dots x_n^{t_n}$$

satisfying the Laplace's equation

$$\Delta P = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} P = 0.$$

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Let $\mathcal{P}_{n,t}$ be the inner product space of all spherical harmonics of degree t on S^{n-1} with

$$\langle P, Q \rangle = \int_{S^{n-1}} P(z) Q(z) \, d\mu_n(z).$$

Riesz representation theorem gives a natural mapping $S^{n-1} \ni x \mapsto P_x \in \mathcal{P}_{n,t}$ such that

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It is well-known that

$$\langle P_x, P_y \rangle = Z_{n,t}(x \cdot y), \quad x, y \in S^{n-1},$$

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$\{C_t^{(\alpha)}(\xi)\}_{t \geq 0}$ are orthogonal on $[-1, 1]$ with the weight $(1 - \xi^2)^{\alpha - 1/2}$, and

$$\frac{1 - z^2}{(1 - 2\xi z + z^2)^{\alpha+1}} = \sum_{t=0}^{\infty} \frac{t + \alpha}{\alpha} C_t^{(\alpha)}(\xi) z^t.$$

For any finite sets of points $\{x_i\}_{i\in\mathcal{I}}$ and $\{y_j\}_{j\in\mathcal{J}}$ from S^{n-1}, using the Cauchy-Shwartz inequality in $\mathcal{P}_{n,t}$, we obtain

$$\begin{aligned}\left(\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}}\langle P_{x_i}, P_{y_j}\rangle\right)^2 &= \left\langle \sum_{i\in\mathcal{I}} P_{x_i}, \sum_{j\in\mathcal{J}} P_{y_j}\right\rangle^2 \\ &\leq \left\langle \sum_{i\in\mathcal{I}} P_{x_i}, \sum_{i\in\mathcal{I}} P_{x_i}\right\rangle \left\langle \sum_{j\in\mathcal{J}} P_{y_j}, \sum_{j\in\mathcal{J}} P_{y_j}\right\rangle \\ &= \sum_{i\in\mathcal{I}}\sum_{i'\in\mathcal{I}}\langle P_{x_i}, P_{x_{i'}}\rangle \sum_{j\in\mathcal{J}}\sum_{j'\in\mathcal{J}}\langle P_{y_j}, P_{y_{j'}}\rangle.\end{aligned}$$

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Rewriting this in terms of $Z_{n,t}$ gives the inequality

$$\left(\sum_{i\in\mathcal{I},j\in\mathcal{J}} Z_{n,t}(x_i\cdot y_j)\right)^2 \le \left(\sum_{i,i'\in\mathcal{I}} Z_{n,t}(x_i\cdot x_{i'})\right)\left(\sum_{j,j'\in\mathcal{J}} Z_{n,t}(y_j\cdot y_{j'})\right). \qquad (*)$$

We take $n := g$,
$\mathcal{I} := V$, $x_i \in S^{n-1} \subset \mathbb{R}^g$ be the Euclidean representation of i, and
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Positive linear combinations of Gegenbauer polynomials are essentially our only choices of zonal functions (Schoenberg).

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- projections of Euclidean representation;
- various combinatorial counting arguments;
- two insignificant computer searches.

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