Non-existence of $(76, 30, 8, 14)$ strongly regular graph and some structural tools

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joint work with Andrii Bondarenko and Danylo Radchenko

A finite, undirected, simple graph $G = (V, E)$ with vertices V and edges E is strongly regular with parameters (v, k, λ, μ) if:

- $|V| = v$,
- G is k-regular,
- any two adjacent vertices have λ common neighbors, and
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In terms of incidence matrix A:

$$AJ = kJ, \quad \text{and} \quad A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J,$$

where I is the identity matrix and J is the matrix with all entries equal to 1.

Relation

$$(v-k-1)\mu = k(k-\lambda-1)$$

is immediate. A has only three eigenvalues:

$$\begin{aligned}
k & \quad \text{of multiplicity } 1, \\
r = \frac{1}{2}\left(\lambda-\mu+\sqrt{(\lambda-\mu)^2+4(k-\mu)}\right) & \quad \text{of multiplicity } f = \frac{1}{2}\left(v-1-\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right), \\
s = \frac{1}{2}\left(\lambda-\mu-\sqrt{(\lambda-\mu)^2+4(k-\mu)}\right) & \quad \text{of multiplicity } g = \frac{1}{2}\left(v-1+\frac{2k+(v-1)(\lambda-\mu)}{\sqrt{(\lambda-\mu)^2+4(k-\mu)}}\right).
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The above relation and the condition that f and g have to be non-negative integers define a family of parameters (v, k, λ, μ) for which the corresponding SRG might exist.

For $(v, k, \lambda, \mu) = (76, 30, 8, 14)$, we have $r^f = 2^{57}$ and $s^g = (-8)^{18}$.

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Unknown cases for small v:
$(65, 32, 15, 16)$,
$(69, 20, 7, 5)$,
$(75, 32, 10, 16)$,
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Note that inverting edges leads to a related SRG.

Outline of the proof:

- K_4 is a subgraph of $SRG(76, 30, 8, 14)$;
- $SRG(76, 30, 8, 14)$ contains (as an induced subgraph) one of the three "larger" subgraphs;
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Techniques:

- Euclidean representation of SRGs - systems of points on the sphere;
- positive definiteness (and rank) of Gram matrix;
- projections of Euclidean representation;
- Cauchy-Shwartz in the space of spherical harmonics;
- various counting arguments, including "frequencies" lemma;
- two insignificant computer searches.

Euclidean representation $\{x_i : i \in V\}$ of a SRG G in $\mathbb{R}^g$:
$A - sI$ is positive semidefinite, therefore $A - sI = (z_i \cdot z_j)_{i,j \in V}$ for some $\{z_i : i \in V\} \subset \mathbb{R}^g$; define $x_i := z_i / \|z_i\|$, $i \in V$.

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$$x_i \cdot x_j = \begin{cases} 1, & \text{if } i = j, \\ p, & \text{if } i \text{ and } j \text{ are adjacent,} \\ q, & \text{otherwise,} \end{cases}$$

where $p = s/k$, and $q = -(s+1)/(v-k-1)$.

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The set $\{x_i : i \in V\}$ forms a spherical 2-design, i.e.,

$$\sum_{i \in V} x_i = 0, \quad \text{and} \quad \sum_{i \in V} (x_i \cdot y)^2 = \frac{|V|}{g} \quad \text{for any } y, \ \|y\| = 1.$$

For $(v,k,\lambda,\mu) = (76,30,8,14)$, we have $r^f = 2^{57}$ and $s^g = (-8)^{18}$.
In $\mathbb{R}^{18}$, $(p,q) = (-\frac{4}{15}, \frac{7}{45})$.
In $\mathbb{R}^{57}$, $(p,q) = (\frac{1}{15}, -\frac{1}{15})$ (edge inversion).

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Otherwise, if $x_1, \ldots, x_5$ are the points in $\mathbb{R}^{18}$ corresponding to K_5, then $0 \leq (x_1 + \cdots + x_5)^2 = 5 + 20p = -1/3$, contradiction.

Towards the lower bound on the number of 4-cliques in a SRG:
Spherical harmonics

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The vector space of all spherical harmonics of degree t on S^{n-1} will be denoted by $\mathcal{P}_{n,t}$.

We can equip $\mathcal{P}_{n,t}$ with the inner product

$$\langle P, Q \rangle = \int_{S^{n-1}} P(x)Q(x)\, d\mu_n(x),$$

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There is a representation through zonal harmonic $Z_{n,t}$:

$$\langle P_x, P_y\rangle = Z_{n,t}(x\cdot y), \quad x, y \in S^{n-1}.$$

We have $Z_{n,t}(\xi) = \frac{2t+n-2}{n-2} C_t^{((n-2)/2)}(\xi)$, where $C_t^{(\alpha)}(\xi)$ are the Gegenbauer polynomials.

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The polynomials $C_t^{(\alpha)}(\xi)$ of degree t are orthogonal on $[-1, 1]$ with the weight $(1-\xi^2)^{\alpha-1/2}$, and can be defined (among other ways) from the generating function

$$\frac{1-z^2}{(1-2\xi z+z^2)^{\alpha+1}} = \sum_{t=0}^{\infty} \frac{t+\alpha}{\alpha} C_t^{(\alpha)}(\xi) z^t.$$

Using the Cauchy-Shwartz inequality in $\mathcal{P}_{n,t}$, for any finite sets of points $\{x_i\}_{i\in\mathcal{I}}$ and $\{y_j\}_{j\in\mathcal{J}}$ from S^{n-1}, we obtain

$$\begin{aligned}\left(\sum_{i\in\mathcal{I},j\in\mathcal{J}}\langle P_{x_i},P_{y_j}\rangle\right)^2 &= \left\langle\sum_{i\in\mathcal{I}}P_{x_i},\sum_{j\in\mathcal{J}}P_{y_j}\right\rangle^2\\ &\le \left\langle\sum_{i\in\mathcal{I}}P_{x_i},\sum_{i\in\mathcal{I}}P_{x_i}\right\rangle\left\langle\sum_{j\in\mathcal{J}}P_{y_j},\sum_{j\in\mathcal{J}}P_{y_j}\right\rangle\\ &= \sum_{i,i'\in\mathcal{I}}\langle P_{x_i},P_{x_{i'}}\rangle\sum_{j,j'\in\mathcal{J}}\langle P_{y_j},P_{y_{j'}}\rangle.\end{aligned}$$

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Rewriting this in terms of $Z_{n,t}$ provides the key inequality

$$\left(\sum_{i\in\mathcal{I},j\in\mathcal{J}}Z_{n,t}(x_i\cdot y_j)\right)^2\leq\left(\sum_{i,i'\in\mathcal{I}}Z_{n,t}(x_i\cdot x_{i'})\right)\left(\sum_{j,j'\in\mathcal{J}}Z_{n,t}(y_j\cdot y_{j'})\right). \qquad (1)$$

With $n = g$, we choose $x_i \in \mathbb{R}^g$ to be the Euclidean representation of $i \in V = \mathcal{I}$, and $y_j := \frac{x_{j^{(1)}} + x_{j^{(2)}}}{\|x_{j^{(1)}} + x_{j^{(2)}}\|}$ for all edges $j \in E = \mathcal{J}$, here j joins the vertices $j^{(1)}, j^{(2)} \in V$.

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It is not hard to prove non-existence of SRG $(460, 153, 32, 60)$ using our bound (work in preparation).

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Positive linear combinations of Gegenbauer polynomials are essentially our only choices of zonal functions (Schoenberg).

Outline of the proof (reminder):

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“Frequencies” lemma.
Let H be a subgraph of G, $m = |H|$, define

$$d_j := |\{x \in H : \text{there are exactly } j \text{ edges from } x \text{ to vertices in } H\}|$$
$$b_j := |\{x \in G \setminus H : \text{there are exactly } j \text{ edges from } x \text{ to vertices in } H\}|.$$

Then

$$\sum_{j\geq 0} b_j = v - m,$$
$$\sum_{j\geq 0} jb_j = mk - \sum_{j\geq 0} jd_j, \quad \text{and}$$
$$\sum_{j\geq 0} \binom{j}{2} b_j = \binom{m}{2}\mu - \sum_{j\geq 0} \binom{j}{2} d_j + \frac{1}{2}(\lambda - \mu) \sum_{j\geq 0} jd_j.$$

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If G is SRG $(76, 30, 8, 14)$ and $H = K_4$, $(d_j)_{j \geq 0} = (0, 0, 0, 4, 0, 0, \dots)$, $b_4 = 0$, and the above system provides $(b_j)_{j \geq 0} = (0, 36, 36, 0, 0, 0 \dots)$.

Setting $G_0 = K_4$, we can partition G into $\{G_0, G_1, G_2\}$, where G_j is the subgraph of $G \setminus G_0$ with vertices connected to exactly j vertices of G_0.
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By strong regularity, $\{G_0, G_1, G_2\}$ is an equitable partition of G with degree matrix

$$\mathcal{D}_{\{G_0, G_1, G_2\}} = \begin{pmatrix} 3 & 1 & 2 \\ 9 & 11 & 18 \\ 18 & 18 & 10 \end{pmatrix}.$$

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Case 2: G_2 has a triangle G_3 — leads to a 16-coclique or to a $K_{6,10}$ (harder, various subcases need to be considered depending on the edges between G_3 and G_0; application of the "frequency" lemma for $H = G_0 \cup G_3$ is the first step in each subcase).

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Case 1: $\widetilde{G} := G_0 \cup G_1$ will be the required $SRG(40, 12, 2, 4)$.

For $x \in \widetilde{G}$ let $H_x = G_2 \cap N(x)$, then $|H_x| = 18$.

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As G_2 has no triangles, by strong regularity each edge of G_2 belongs to exactly 8 triangles, where all 8 "third" vertices belong to $\widetilde{G}$.

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As G_2 has no triangles, by strong regularity each edge of G_2 belongs to exactly 8 triangles, where all 8 "third" vertices belong to $\widetilde{G}$.
The average number of edges in H_x over all $x \in \widetilde{G}$ is precisely $\frac{180 \cdot 8}{40} = 36$.

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Suppose $x \in G_1$. Consider the partition $\pi = \{G_0, H_x, \{x\}\}$ of $4+18+1 = 23$ vertices of G.
The quantities of edges in and between the corresponding subgraphs form the edge matrix of the partition

$$\mathcal{E}_\pi = \begin{pmatrix} 6 & 2 \cdot 18 & 1 \\ & w & 18 \\ & & 0 \end{pmatrix}.$$

Let $\pi = \{V_1, \ldots, V_l\}$ be a partition of a subset $\widetilde{V} \subset V$ of the vertices of a (v, k, λ, μ) SRG graph $G = (V, E)$.
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Set $X_j := \sum_{i \in V_j} x_i$, $j = 1, \dots, l$, and let $M(\pi, p, q) = (M_{i,j})$ be the Gram matrix of X_j, i.e., $M_{i,j} := X_i \cdot X_j$, $i, j = 1, \dots, l$.

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$$M_{i,i} = m_i + 2a_{i,i}p + (m_i(m_i - 1) - 2a_{i,i})q, \quad M_{i,j} = a_{i,j}p + (m_i m_j - a_{i,j})q.$$

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For our situation, $\det M(\pi, p, q) = \frac{722}{1125}(36 - w) \geq 0$, so $w \leq 36$.

If $\det M(\pi, p, q) = 0$, the vectors X_j are linearly dependent and for some λ_j we obtain $\sum_j \lambda_j X_j = 0$. For any vertex $z \in G$, let e_j be the number of neighbors of z in G_j. Then $x_z \cdot (\sum_j \lambda_j X_j) = 0$ becomes

$$\sum_{j:z \notin V_j} \lambda_j(pe_j + q(|V_j| - e_j)) + \sum_{j:z \in V_j} \lambda_j(1 + pe_j + q(|V_j| - 1 - e_j)) = 0.$$

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This establishes that $\widetilde{G}$ is $SRG(40, 12, 2, 4)$.

One can prove not only that $\widetilde{G}$ is a $SRG(40, 12, 2, 4)$, but also that for any $z \in G \setminus \widetilde{G}$ both $N(z) \cap \widetilde{G}$ and $N'(z) \cap \widetilde{G}$ are 4-regular subgraphs with 20 vertices, and $|N(z_1) \cap N(z_2) \cap \widetilde{G}| = 8$ for any adjacent $z_1, z_2 \in G \setminus \widetilde{G}$.

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For $j^{(1)}, j^{(2)} \in G \setminus \widetilde{G}$, the goal is to compute the dot product $x''_{j^{(1)}} \cdot x''_{j^{(2)}}$, where $x''_j = x_j - x'_j$ is the projection of x_j onto the orthogonal complement of $\mathrm{lin}\{x_i, i \in \widetilde{G}\}$, which is a $18 - 16 = 2$-dimensional Euclidean space.

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We obtain

$$x'_j = -\frac{1}{9} \sum_{i \in N(j) \cap \widetilde{G}} x_i + \frac{1}{18} \sum_{i \in N'(j) \cap \widetilde{G}} x_i.$$

With $n_{j^{(1)},j^{(2)}} := |N(j^{(1)}) \cap N(j^{(2)}) \cap \widetilde{G}|$, we can prove that

$$x'_{j^{(1)}} \cdot x'_{j^{(2)}} = \frac{19}{270} n_{j^{(1)},j^{(2)}} - \frac{52}{81}.$$

Our construction yields $n_{j^{(1)},j^{(2)}} = 20$ if $j^{(1)} = j^{(2)}$, so all $\|x'_j\|$ are equal ($j \in G \setminus \widetilde{G}$), and hence all $\|x''_j\|$ are equal. This means that all x''_j belong to a (2-dimensional, planar) circle.

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The normalized projections $x'''_j := \frac{x''_j}{\|x''_j\|}$ can be computed by

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In particular, if $j^{(1)}$ and $j^{(2)}$ are adjacent, $n_{j^{(1)},j^{(2)}} = 8$, so $x'''_{j^{(1)}} \cdot x'''_{j^{(2)}} = -\frac{4}{5}$.

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If $(-\frac{4}{5}, -\frac{3}{5})$ is also attained, then for the corresponding vertices $x'''_{j^{(1)}} \cdot x'''_{j^{(2)}} = \frac{7}{25}$ leading to $n_{j^{(1)},j^{(2)}} = \frac{52}{5}$.

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But then clearly

$$\sum_{i \in G \setminus \widetilde{G}} x_i''' \neq (0,0).$$

On the other hand, $\sum_{i \in G} x_i = 0$ and $x_i'' = (0,0)$ for $i \in \widetilde{G}$, so

$$\sum_{i \in G \setminus \widetilde{G}} x_i'' = (0,0),$$

which is a contradiction completing the case of $SRG(40, 12, 2, 4)$.

If G contains a 16-coclique $\widetilde{G}$, we establish $x'''_{j^{(1)}} \cdot x'''_{j^{(2)}} \in \{1, -\frac{1}{2}\}$ on the unit circle. WLOG, $x'''_j = (\cos(t\pi/3), \sin(t\pi/3))$, $j \in H_t$, $t = 1, 2, 3$, where $G \setminus \widetilde{G} = H_1 \cup H_2 \cup H_3$. Then $\sum_{j \in G \setminus \widetilde{G}} x'''_j = (0, 0)$, which implies $|H_1| = |H_2| = |H_3| = 20$.

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Next we establish that H_1 is 2-regular, so it is a union of cycles. Further analysis yields that there are at least four cycles in H_1, and each of them has an even length. Hence, there is a cycle $C_4 \subset H_1$ of length 4.

Suppose that $\widetilde{G} = \{g_1, \ldots, g_{16}\}$. For $i \in H_1$, define $A(i)$ as the 8-element subset of $\{1, 2, \ldots, 16\}$ such that $N(i) \cap \widetilde{G} = \{g_t : t \in A(i)\}$.

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$$\{A(i) : i \in C_4\} = \{\{1,2,3,4,5,6,7,8\}, \{1,2,9,10,11,12,13,14\}, \{5,6,7,8,13,14,15,16\}, \{3,4,9,10,11,12,15,16\}\}.$$

Let $\mathfrak{M}$ be the collection of all 8-element subsets of $\{1, 2, \ldots, 16\}$, $|\mathfrak{M}| = \binom{16}{8} = 12870$.
Define the following graph on $\mathfrak{M}$:
two vertices (subsets) $A_1, A_2 \in \mathfrak{M}$ are adjacent iff $|A_1 \cap A_2| \in \{2, 4\}$.
With $\mathfrak{M}_0 := \{A(i) : i \in C_4\}$, $|\mathfrak{M}_0| = 4$, let
$\mathfrak{M}_1 := \{A \in \mathfrak{M} : A$ is adjacent to all vertices of $\mathfrak{M}_0\}$. Then $|\mathfrak{M}_1| = 906$ and this subgraph will have 176672 edges.

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Clearly, $\{A(i) : i \in H_1 \setminus C_4\}$ is a 16-clique in $\mathfrak{M}_1$.

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$\mathfrak{M}_1 := \{A \in \mathfrak{M} : A \text{ is adjacent to all vertices of } \mathfrak{M}_0\}$. Then $|\mathfrak{M}_1| = 906$ and this subgraph will have 176672 edges.
Clearly, $\{A(i) : i \in H_1 \setminus C_4\}$ is a 16-clique in $\mathfrak{M}_1$.
But the largest clique in $\mathfrak{M}_1$ has size 15, which can be verified using `clique_number` function of Sage based on the Bron-Kerbosch algorithm.

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The main idea for the completion of the proof is to verify (with an assistance of a computer algebra system) that all such subgraphs $U = H_1 \cup H_2$ fail to satisfy the following statement:

Each subset $\{x_i : i \in U\}$, where $U \subset V$, has a non-negative definite Gram matrix $(x_i \cdot x_j)_{i,j \in U}$ of rank equal to the rank of the linear span of $\{x_i : i \in U\}$. If A is the adjacency matrix of the subgraph induced by U, then $(x_i \cdot x_j)_{i,j \in U} = pA + I + q(J - I - A)$.

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