Covering problems in high dimensions: Borsuk's question and Lebesgue's universal cover

Andrii Arman and Andriy Prymak
(joint work with A. Bondarenko and D. Radchenko)

University of Manitoba

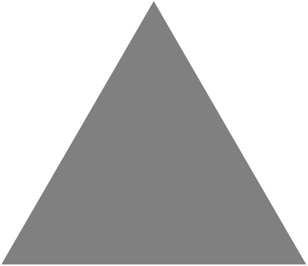
Borsuk's number

Let $K \subset \mathbb{R}^n$, $\mathrm{diam}(K) = \sup_{x,y\ \in K} \|x - y\|$.

Definition

For $K \subset \mathbb{R}^n$ with $\mathrm{diam}(K) = d$, let $b(K)$ be the smallest integer such that K can be covered by $b(K)$ sets of a diameter $< d$.

Define $\mathbf{b(n)} = \sup\{b(K)\ :\ K \subset \mathbb{R}^n,\ \mathrm{diam}(K) = 1\}$.



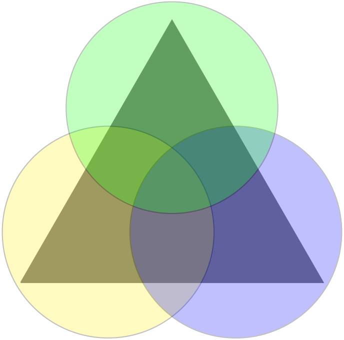
Borsuk's number

Definition

For $K \subset \mathbb{R}^n$ with $\operatorname{diam}(K) = d$, let $b(K)$ be the smallest integer such that K can be covered by $b(K)$ sets of a diameter $< d$.

Define $\mathbf{b(n)} = \sup\{b(K) \;:\; K \subset \mathbb{R}^n,\ \operatorname{diam}(K) = 1\}$.

Approach #1: Grünbaum (use balls).



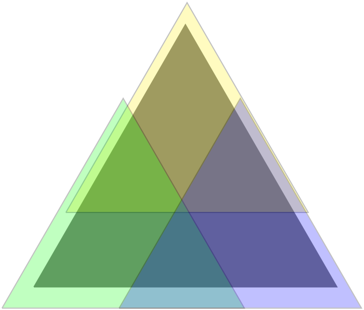
Borsuk's number

Definition

For $K \subset \mathbb{R}^n$ with $\operatorname{diam}(K) = d$, let $b(K)$ be the smallest integer such that K can be covered by $b(K)$ sets of a diameter $< d$.

Define $\mathbf{b(n)} = \sup\{b(K) \ : \ K \subset \mathbb{R}^n, \operatorname{diam}(K) = 1\}$.

Approach #2: Schramm (use smaller copies of K).



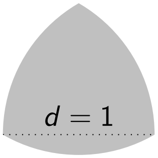
Borsuk's number

Definition

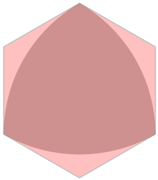
For $K \subset \mathbb{R}^n$ with $\operatorname{diam}(K) = d$, let $b(K)$ be the smallest integer such that K can be covered by $b(K)$ sets of a diameter $< d$.

Define $\mathbf{b(n)} = \sup\{b(K) \;:\; K \subset \mathbb{R}^n,\ \operatorname{diam}(K) = 1\}$.

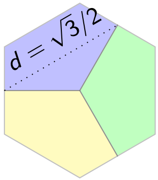
Approach #3: Use universal covers.



K



U



U

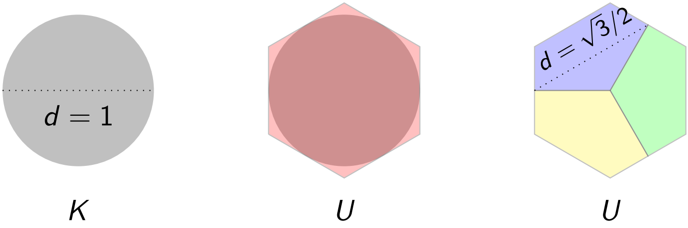
Borsuk's number

Definition

For $K \subset \mathbb{R}^n$ with $\mathrm{diam}(K) = d$, let $b(K)$ be the smallest integer such that K can be covered by $b(K)$ sets of a diameter $< d$.

Define $\mathbf{b(n)} = \sup\{b(K) \;:\; K \subset \mathbb{R}^n,\ \mathrm{diam}(K) = 1\}$.

Approach #3: Use universal covers.



Borsuk's number

Definition

For $K \subset \mathbb{R}^n$ with $\mathrm{diam}(K) = d$, let $b(K)$ be the smallest integer such that K can be covered by $b(K)$ sets of a diameter $< d$.

Define $\mathbf{b(n)} = \sup\{b(K) \ : \ K \subset \mathbb{R}^n, \ \mathrm{diam}(K) = 1\}$.

Borsuk 1933: $b(1) = 2$, $b(2) = 3$, $b(\mathbb{S}^{n-1}) = n + 1$.

Borsuk's number

Definition

For $K \subset \mathbb{R}^n$ with $\mathrm{diam}(K) = d$, let $b(K)$ be the smallest integer such that K can be covered by $b(K)$ sets of a diameter $< d$.

Define $\mathbf{b(n)} = \sup\{b(K) \ : \ K \subset \mathbb{R}^n, \ \mathrm{diam}(K) = 1\}$.

Borsuk 1933: $b(1) = 2$, $b(2) = 3$, $b(\mathbb{S}^{n-1}) = n + 1$.

Question (Borsuk 1933)

Is it true that for all n, $b(n) = n + 1$?

Borsuk's number

Definition

For $K \subset \mathbb{R}^n$ with $\operatorname{diam}(K) = d$, let $b(K)$ be the smallest integer such that K can be covered by $b(K)$ sets of a diameter $< d$.

Define $\mathbf{b(n)} = \sup\{b(K) \ : \ K \subset \mathbb{R}^n, \ \operatorname{diam}(K) = 1\}$.

Borsuk 1933: $b(1) = 2$, $b(2) = 3$, $b(\mathbb{S}^{n-1}) = n + 1$.

Question (Borsuk 1933)

Is it true that for all n, $b(n) = n + 1$?

Perkal 1947: $b(3) = 4$.

$b(n) \neq n+1$

Kahn, Kalai 1993: $b(n) \geq 1.2^{\sqrt{n}}$ for large n.

$b(n) \neq n+1$

Kahn, Kalai 1993: $b(n) \geq 1.2^{\sqrt{n}}$ for large n.
Raigorodskii 1999: $b(n) \geq (1.2255)^{\sqrt{n}}$ for large n

$b(n) \neq n+1$

Kahn, Kalai 1993: $b(n) \geq 1.2^{\sqrt{n}}$ for large n.
Raigorodskii 1999: $b(n) \geq (1.2255)^{\sqrt{n}}$ for large n

Question

For which values of n do we have $b(n) > n+1$?

$b(n) \neq n + 1$

Kahn, Kalai 1993: $b(n) \geq 1.2^{\sqrt{n}}$ for large n.
Raigorodskii 1999: $b(n) \geq (1.2255)^{\sqrt{n}}$ for large n

Question

For which values of n do we have $b(n) > n + 1$?

Kahn, Kalai 1993:	$n \geq 2015$, $n = 1325$,
Raigorodskii 1997:	$n = 561$,
Raigorodskii 1999:	$n \geq 561$,
Hinrichs 2002:	$n \geq 323$,
Pikhurko 2002:	$n \geq 321$,
Hinrichs, Richter 2003:	$n \geq 298$,
Bondarenko 2013:	$n \geq 65$,
Jenrich 2013:	$n \geq 64$.

Upper bounds on $b(n)$

Rogers 1965: $b(n) \le \left(\sqrt{2} + o(1)\right)^n$.
K with $\mathrm{diam}(K) = 1$ is a subset of a ball of diameter $\sqrt{2}$. Covered big ball with open balls of diameter 1.

Lassak 1982: $b(n) \le 2^{n-1} + 1$.
K is in intersection of two balls of diameters $\sqrt{\frac{2n}{n+1}}$ and 2, used octants + cap.

Schramm 1988: $b(n) \le \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$.
K is a body of constant width 1, covered K by smaller translates of K.

Bourgain Lindenstrauss 1989: $b(n) \le \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$.
Covered K by balls of diameter less than 1.

Upper bounds on $b(n)$

Rogers 1965: $b(n) \leq \left(\sqrt{2} + o(1)\right)^n$.
K with $\mathrm{diam}(K) = 1$ is a subset of a ball of diameter $\sqrt{2}$. Covered big ball with open balls of diameter 1.

Lassak 1982: $b(n) \leq 2^{n-1} + 1$.
K is in intersection of two balls of diameters $\sqrt{\frac{2n}{n+1}}$ and 2, used octants + cap.

Schramm 1988: $b(n) \leq \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$.
K is a body of constant width 1, covered K by smaller translates of K.

Bourgain Lindenstrauss 1989: $b(n) \leq \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$.
Covered K by balls of diameter less than 1.

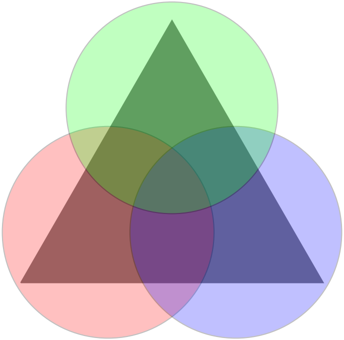
Question (Kalai 2015)

Is there a $C > 1$, such that $b(n) > C^n$ for all n.

Bourgain Lindenstrauss approach

Definition (Grünbaum 1950s)

Let $\mathrm{diam}(K) = d$, define $g(K)$ be the smallest number of balls of diameter $< d$ needed to cover K. Let $\mathbf{g(n)} = \sup\{g(K) : K \subset \mathbb{R}^n, \mathrm{diam}(K) = 1\}$.



Bourgain Lindenstrauss approach

Definition (Grünbaum 1950s)

Let $\mathrm{diam}(K) = d$, define $g(K)$ be the smallest number of balls of diameter $< d$ needed to cover K. Let $\mathbf{g(n)} = \sup\{g(K) : K \subset \mathbb{R}^n, \mathrm{diam}(K) = 1\}$.

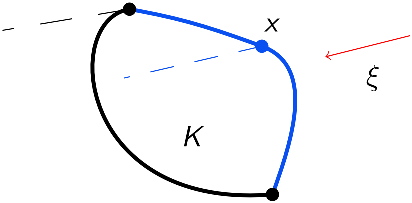
Note that $b(n) \leq g(n)$.

Danzer 1965: $g(n) \geq (1.003)^n$
Rogers 1965: $g(n) \leq (\sqrt{2} + o(1))^n$

Bourgain Lindenstrauss 1989: $(1.0645)^n \leq g(n) \leq \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$.

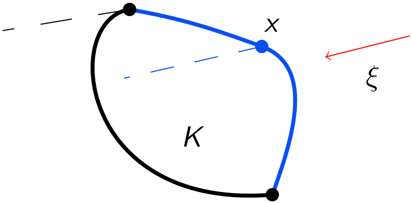
Schramm's approach – illumination and covering

Let K be a convex body in $\mathbb{E}^n$. A point $x \in \partial K$ is **illuminated** by a direction $\xi \in \mathbb{S}^{n-1}$ if the ray $\{x + \xi t : t \geq 0\}$ intersects $\text{int}(K)$.



Schramm's approach – illumination and covering

Let K be a convex body in $\mathbb{E}^n$. A point $x \in \partial K$ is **illuminated** by a direction $\xi \in \mathbb{S}^{n-1}$ if the ray $\{x + \xi t : t \geq 0\}$ intersects $\mathrm{int}(K)$.



The illumination number $I(K)$ is the minimal number of directions such that every $x \in \partial K$ is illuminated by one of these directions.

Schramm's approach – illumination and covering

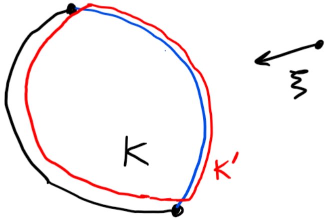
The illumination number $I(K)$ is the minimal number of directions such that every $x \in \partial K$ is illuminated by one of these directions.

Boltyanski (1960): $I(K)$ is equal to the minimum number of smaller positive homothetic translates of K needed to cover K.

Schramm's approach – illumination and covering

The illumination number $I(K)$ is the minimal number of directions such that every $x \in \partial K$ is illuminated by one of these directions.

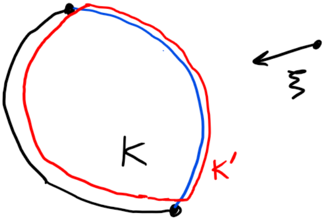
Boltyanski (1960): $I(K)$ is equal to the minimum number of smaller positive homothetic translates of K needed to cover K.



Schramm's approach – illumination and covering

The illumination number $I(K)$ is the minimal number of directions such that every $x \in \partial K$ is illuminated by one of these directions.

Boltyanski (1960): $I(K)$ is equal to the minimum number of smaller positive homothetic translates of K needed to cover K.



Conjecture (Levi-Hadwiger-Gohberg-Markus's)

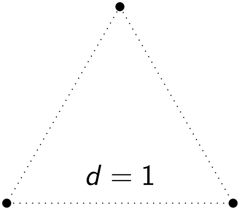
$I(K) \leq 2^n$ with equality iff K is an affine copy of a cube.

Schramm's approach – bodies of constant width

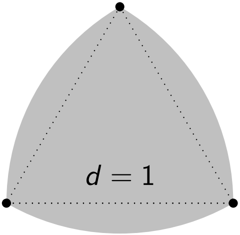
Eggleston 1958: Any set of diameter d is contained in a convex body of constant width d.

Schramm's approach – bodies of constant width

Eggleston 1958: Any set of diameter d is contained in a convex body of constant width d.



K



W

Therefore, it suffices to consider only bodies of constant width when computing the Borsuk's number $b(n)$.

Schramm's upper bound on Borsuk's number

Definition

$\mathbf{h(n)} := \sup\{I(K) : K \text{ is a convex body of constant width in } \mathbb{E}^n\}$.

We have $b(n) \leq h(n)$.

Schramm's upper bound on Borsuk's number

Definition

$\mathbf{h(n)} := \sup\{I(K) : K \text{ is a convex body of constant width in } \mathbb{E}^n\}$.

We have $b(n) \leq h(n)$.

Schramm 1988: $h(n) \leq \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$.

Schramm's upper bound on Borsuk's number

Definition

$\mathbf{h(n)} := \sup\{I(K) : K$ is a convex body of constant width in $\mathbb{E}^n\}$.

We have $b(n) \leq h(n)$.

Schramm 1988: $h(n) \leq \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$.

Question (Kalai 2015)

Does there exists $C > 1$ such that $h(n) \geq C^n$?

Question (Kalai 2015)

Does there exists $C > 1$ such that $h(n) \geq C^n$?

Bourgain Lindenstrauss 1989: $(1.0645...)^n \leq g(n) \leq \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$.

Theorem (A., Bondarenko, Prymak 2023)

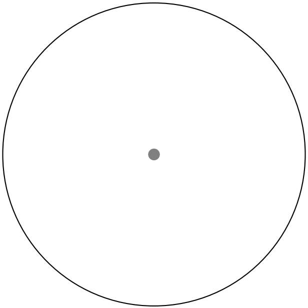
For some constants c_1, c_2,

$$h(n) \geq \frac{c_1}{\sqrt{n}} \left(\frac{1}{\cos\left(\frac{\pi}{14}\right)}\right)^n \approx (1.0257..)^n,$$

$$g(n) \geq \frac{c_2}{\sqrt{n}} \left(\frac{2}{\sqrt{3}}\right)^n \approx (1.1547..)^n.$$

Definition (Cap)

$C(x, \varphi) = \{y \in \mathbb{S}^{n-1} : \theta(x, y) \leq \varphi\}$ for $x \in \mathbb{S}^{n-1}$, $\varphi \in [0, \pi/2]$.



Definition (Cap)

$C(x,\varphi) = \{y \in \mathbb{S}^{n-1} : \theta(x,y) \le \varphi\}$ for $x \in \mathbb{S}^{n-1}$, $\varphi \in [0,\pi/2]$.

Folklore:
Let μ be the normalized surface area on $\mathbb{S}^{n-1}$. If $\varphi \in [0,\pi/2)$ is fixed, then for large n

$$\mu(C(x,\varphi)) = \frac{\Theta(1)}{\sqrt{n}} (\sin(\varphi))^n \,.$$

Notation: $\mu(\varphi) = \mu(C(x,\varphi))$.

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq 12n \log n$ *for all* $x \in \mathbb{S}^{n-1}$.

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists* $X = \{x_1, \dots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq 12 n \log n$ *for all* $x \in \mathbb{S}^{n-1}$.

Proof idea (b):

1. For $M = \frac{4n \log n}{\mu(\varphi)}$ let $Y = \{x_1, \dots, x_M\}$ be i.i.d. with $x_i \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x,\varphi) \cap X| \leq 12n\log n$ *for all* $x \in \mathbb{S}^{n-1}$.

Proof idea (b):

1. For $M = \frac{4n\log n}{\mu(\varphi)}$ let $Y = \{x_1, \ldots, x_M\}$ be i.i.d. with $x_i \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.
2. For $\varepsilon = n^{-2}$ let $\mathcal{N}$ be an ε-net on $\mathbb{S}^{n-1}$, $|\mathcal{N}| \leq (n^2 + o(1))^n \leq e^{3n\log n}$

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq 12n \log n$ *for all* $x \in \mathbb{S}^{n-1}$.

Proof idea (b):

1. For $M = \frac{4n\log n}{\mu(\varphi)}$ let $Y = \{x_1, \ldots, x_M\}$ be i.i.d. with $x_i \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.
2. For $\varepsilon = n^{-2}$ let $\mathcal{N}$ be an ε-net on $\mathbb{S}^{n-1}$, $|\mathcal{N}| \leq (n^2 + o(1))^n \leq e^{3n\log n}$
3. For every $x \in \mathbb{S}^{n-1}$ exists $y \in \mathcal{N}$ such that $C(x, \varphi) \subseteq C(y, \varphi + \varepsilon)$.

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq 12n\log n$ *for all* $x \in \mathbb{S}^{n-1}$.

Proof idea (b):

1. For $M = \frac{4n\log n}{\mu(\varphi)}$ let $Y = \{x_1, \ldots, x_M\}$ be i.i.d. with $x_i \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.
2. For $\varepsilon = n^{-2}$ let $\mathcal{N}$ be an ε-net on $\mathbb{S}^{n-1}$, $|\mathcal{N}| \leq (n^2 + o(1))^n \leq e^{3n\log n}$
3. For every $x \in \mathbb{S}^{n-1}$ exists $y \in \mathcal{N}$ such that $C(x, \varphi) \subseteq C(y, \varphi + \varepsilon)$.
4. For all $y \in \mathbb{S}^{n-1}$, $|Y \cap C(y, \varphi + \varepsilon)| \sim B(M, \mu(\varphi + \varepsilon))$.

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x,\varphi) \cap X| \leq 12n\log n$ *for all* $x \in \mathbb{S}^{n-1}$.

Proof idea (b):

1. For $M = \frac{4n\log n}{\mu(\varphi)}$ let $Y = \{x_1, \ldots, x_M\}$ be i.i.d. with $x_i \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.
2. For $\varepsilon = n^{-2}$ let $\mathcal{N}$ be an ε-net on $\mathbb{S}^{n-1}$, $|\mathcal{N}| \leq (n^2 + o(1))^n \leq e^{3n\log n}$
3. For every $x \in \mathbb{S}^{n-1}$ exists $y \in \mathcal{N}$ such that $C(x,\varphi) \subseteq C(y, \varphi + \varepsilon)$.
4. For all $y \in \mathbb{S}^{n-1}$, $|Y \cap C(y, \varphi+\varepsilon)| \sim B(M, \mu(\varphi+\varepsilon))$.
5. Chernoff: $\mathbb{P}(|Y \cap C(y,\varphi+\varepsilon)| \geq 3 \cdot 4n\log n) \leq e^{-4n\log n}$.

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists* $X = \{x_1, \dots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq 12n \log n$ *for all* $x \in \mathbb{S}^{n-1}$.

Proof idea (b):

1. For $M = \frac{4n\log n}{\mu(\varphi)}$ let $Y = \{x_1, \dots, x_M\}$ be i.i.d. with $x_i \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.
2. For $\varepsilon = n^{-2}$ let $\mathcal{N}$ be an ε-net on $\mathbb{S}^{n-1}$, $|\mathcal{N}| \leq (n^2 + o(1))^n \leq e^{3n\log n}$
3. For every $x \in \mathbb{S}^{n-1}$ exists $y \in \mathcal{N}$ such that $C(x, \varphi) \subseteq C(y, \varphi + \varepsilon)$.
4. For all $y \in \mathbb{S}^{n-1}$, $|Y \cap C(y, \varphi + \varepsilon)| \sim B(M, \mu(\varphi + \varepsilon))$.
5. Chernoff: $\mathbb{P}(|Y \cap C(y, \varphi + \varepsilon)| \geq 3 \cdot 4n\log n) \leq e^{-4n\log n}$.
6. Union bound: w.h.p, for all $y \in \mathcal{N}$, $|C(y, \varphi + \varepsilon) \cap X| \leq 12n\log n$.

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$*. Then for any sufficiently large* n *there exists* $X = \{x_1, \dots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$*;*

(b) $|C(x, \varphi) \cap X| \leq 12n \log n$ *for all* $x \in \mathbb{S}^{n-1}$*.*

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq 12n \log n$ *for all* $x \in \mathbb{S}^{n-1}$.

Proof idea (a):

1. For $M = 4\frac{n \log n}{\mu(\varphi)}$ let $Y = \{x_1, \ldots, x_M\}$ be i.i.d. with $x_i \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$*. Then for any sufficiently large* n *there exists* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$*;*

(b) $|C(x, \varphi) \cap X| \leq 12 n \log n$ *for all* $x \in \mathbb{S}^{n-1}$*.*

Proof idea (a):

1. For $M = 4\frac{n \log n}{\mu(\varphi)}$ let $Y = \{x_1, \ldots, x_M\}$ be i.i.d. with $x_i \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.
2. $\mathbb{P}(\{x_i, x_j\} \text{ fails (a)}) = 2\mu(\varphi)$ for all i, j.

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$*. Then for any sufficiently large* n *there exists* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$*;*

(b) $|C(x, \varphi) \cap X| \leq 12n \log n$ *for all* $x \in \mathbb{S}^{n-1}$*.*

Proof idea (a):

1. For $M = 4\frac{n \log n}{\mu(\varphi)}$ let $Y = \{x_1, \ldots, x_M\}$ be i.i.d. with $x_i \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.
2. $\mathbb{P}(\{x_i, x_j\}$ fails (a)$) = 2\mu(\varphi)$ for all i, j.
3. Let $B = B(Y) = \{\{x_i, x_j\} : \theta(x_i, x_j) \notin [\varphi, \pi - \varphi]\}$, and graph $G = (Y, B)$.

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq 12 n \log n$ *for all* $x \in \mathbb{S}^{n-1}$.

Proof idea (a):

1. For $M = 4\frac{n \log n}{\mu(\varphi)}$ let $Y = \{x_1, \ldots, x_M\}$ be i.i.d. with $x_i \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.
2. $\mathbb{P}(\{x_i, x_j\} \text{ fails (a)}) = 2\mu(\varphi)$ for all i, j.
3. Let $B = B(Y) = \{\{x_i, x_j\} : \theta(x_i, x_j) \notin [\varphi, \pi - \varphi]\}$, and graph $G = (Y, B)$.
4. Let $d(G)$ be average degree of G, then $\mathbb{E}(d(G)) < 8 n \log n$.

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists* $X = \{x_1, \dots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq 12n \log n$ *for all* $x \in \mathbb{S}^{n-1}$.

Proof idea (a):

1. For $M = 4\frac{n \log n}{\mu(\varphi)}$ let $Y = \{x_1, \dots, x_M\}$ be i.i.d. with $x_i \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.
2. $\mathbb{P}(\{x_i, x_j\}$ fails (a)$) = 2\mu(\varphi)$ for all i, j.
3. Let $B = B(Y) = \{\{x_i, x_j\} : \theta(x_i, x_j) \notin [\varphi, \pi - \varphi]\}$, and graph $G = (Y, B)$.
4. Let $d(G)$ be average degree of G, then $\mathbb{E}(d(G)) < 8n \log n$.
5. Caro 1979: $\alpha(G) \geq \frac{|V(G)|}{d(G)+1}$. With positive probability $\alpha(G) > \frac{1}{3}\frac{1}{\mu(\varphi)}$.

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$*. Then for any sufficiently large n there exists* $X = \{x_1, \dots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$*;*

(b) $|C(x, \varphi) \cap X| \leq 12n \log n$ *for all* $x \in \mathbb{S}^{n-1}$*.*

Proof idea (a):

1. For $M = 4\frac{n \log n}{\mu(\varphi)}$ let $Y = \{x_1, \dots, x_M\}$ be i.i.d. with $x_i \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.
2. $\mathbb{P}(\{x_i, x_j\}$ fails (a)$) = 2\mu(\varphi)$ for all i, j.
3. Let $B = B(Y) = \{\{x_i, x_j\} : \theta(x_i, x_j) \notin [\varphi, \pi - \varphi]\}$, and graph $G = (Y, B)$.
4. Let $d(G)$ be average degree of G, then $\mathbb{E}(d(G)) < 8n \log n$.
5. Caro 1979: $\alpha(G) \geq \frac{|V(G)|}{d(G)+1}$. With positive probability $\alpha(G) > \frac{1}{3}\frac{1}{\mu(\varphi)}$.
6. There is $X \subset Y$ that satisfies (a), and $|X| > \frac{1}{3}\frac{1}{\mu(\varphi)}$.

Thinly spread codes – weak version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{3\mu(\varphi)} \geq \frac{c\sqrt{n}}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq 12n \log n$ *for all* $x \in \mathbb{S}^{n-1}$.

Proof idea (a):

1. For $M = 4\frac{n \log n}{\mu(\varphi)}$ let $Y = \{x_1, \ldots, x_M\}$ be i.i.d. with $x_i \sim \mathrm{Unif}(\mathbb{S}^{n-1})$.
2. $\mathbb{P}(\{x_i, x_j\}$ fails (a)$) = 2\mu(\varphi)$ for all i, j.
3. Let $B = B(Y) = \{\{x_i, x_j\} : \theta(x_i, x_j) \notin [\varphi, \pi - \varphi]\}$, and graph $G = (Y, B)$.
4. Let $d(G)$ be average degree of G, then $\mathbb{E}(d(G)) < 8n \log n$.
5. Caro 1979: $\alpha(G) \geq \frac{|V(G)|}{d(G)+1}$. With positive probability $\alpha(G) > \frac{1}{3}\frac{1}{\mu(\varphi)}$.
6. There is $X \subset Y$ that satisfies (a), and $|X| > \frac{1}{3}\frac{1}{\mu(\varphi)}$.
7. Finally, since $X \subseteq Y$ still satisfies (b).

Thinly spread codes – a better version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists a collection* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{4}\frac{\log n}{\mu(\varphi)} \geq \frac{c\sqrt{n}\log n}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq 12n\log n$ *for all* $x \in \mathbb{S}^{n-1}$.

Thinly spread codes – a better version of Lemma

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists a collection* $X = \{x_1, \dots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{1}{4}\frac{\log n}{\mu(\varphi)} \geq \frac{c\sqrt{n}\log n}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq 12n\log n$ *for all* $x \in \mathbb{S}^{n-1}$.

Proof idea:

1. $M = 4\frac{n\log n}{\mu(\varphi)}$, $\mathbb{E}(e(G)) \approx (8n\log n)M$, $d(G) \approx 8n\log n$.
2. $t_3(G)$ – number of triangles in G, then $\mathbb{E}(t_3(G)) = o(M/n)$.
3. Remove all triangles (deleting edges) from G to derive $H \subseteq G$ with $V(H) = M$, $d(H) \approx 8n\log n$, and $t_3(H) = 0$.
4. Shearer 1983: In a triangle-free H, $\alpha(H) \geq \frac{|V(H)|\log d(H)}{d(H)}$.
   There is independent set Z in H with $|Z| \geq \frac{1}{2}\frac{\log n}{\mu(\varphi)}$.
5. Delete vertices of triangles from Z to obtain X with $|X| \geq \frac{1}{4}\frac{\log n}{\mu(\varphi)}$.

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists a collection* $X = \{x_1, \dots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{c\sqrt{n}\log n}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq Cn\log n$ *for all* $x \in \mathbb{S}^{n-1}$.

$g(X)$ is min number of balls of diameter $< \operatorname{diam}(X)$ needed to cover X.

Theorem

There is $X \subseteq \mathbb{S}^{n-1}$ *such that* $g(X) \geq \frac{c_2}{\sqrt{n}}\left(\frac{2}{\sqrt{3}}\right)^n$.

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists a collection* $X = \{x_1, \dots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{c\sqrt{n}\log n}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$*;*

(b) $|C(x, \varphi) \cap X| \leq Cn\log n$ *for all* $x \in \mathbb{S}^{n-1}$.

$g(X)$ is min number of balls of diameter $< \operatorname{diam}(X)$ needed to cover X.

Theorem

There is $X \subseteq \mathbb{S}^{n-1}$ *such that* $g(X) \geq \frac{c_2}{\sqrt{n}}\left(\frac{2}{\sqrt{3}}\right)^n$.

Proof:

1. Use Lemma 1 with $\varphi = \frac{\pi}{3}$ to get a thinly spread $X \subset \mathbb{S}^{n-1}$.

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists a collection* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{c\sqrt{n}\log n}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq Cn \log n$ *for all* $x \in \mathbb{S}^{n-1}$.

$g(X)$ is min number of balls of diameter $< \operatorname{diam}(X)$ needed to cover X.

Theorem

There is $X \subseteq \mathbb{S}^{n-1}$ *such that* $g(X) \geq \frac{c_2}{\sqrt{n}} \left(\frac{2}{\sqrt{3}}\right)^n$.

Proof:

1. Use Lemma 1 with $\varphi = \frac{\pi}{3}$ to get a thinly spread $X \subset \mathbb{S}^{n-1}$.
2. From (a), $\operatorname{diam} X \leq 2\sin(\pi/3) = \sqrt{3}$.

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists a collection* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{c\sqrt{n}\log n}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq Cn \log n$ *for all* $x \in \mathbb{S}^{n-1}$.

$g(X)$ is min number of balls of diameter $< \operatorname{diam}(X)$ needed to cover X.

Theorem

There is $X \subseteq \mathbb{S}^{n-1}$ *such that* $g(X) \geq \frac{c_2}{\sqrt{n}} \left(\frac{2}{\sqrt{3}}\right)^n$.

Proof:

1. Use Lemma 1 with $\varphi = \frac{\pi}{3}$ to get a thinly spread $X \subset \mathbb{S}^{n-1}$.
2. From (a), $\operatorname{diam} X \leq 2\sin(\pi/3) = \sqrt{3}$.
3. Any ball of diameter $\sqrt{3}$ intersects $\mathbb{S}^{n-1}$ by a cap of radius $\leq \pi/3$.

Lemma 1

Suppose $0 < \varphi < \frac{\pi}{2}$. *Then for any sufficiently large* n *there exists a collection* $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ *with* $N \geq \frac{c\sqrt{n}\log n}{(\sin\varphi)^n}$ *such that*

(a) $\varphi \leq \theta(x_i, x_j) \leq \pi - \varphi$ *for all* $i \neq j$;

(b) $|C(x, \varphi) \cap X| \leq Cn\log n$ *for all* $x \in \mathbb{S}^{n-1}$.

$g(X)$ is min number of balls of diameter $< \operatorname{diam}(X)$ needed to cover X.

Theorem

There is $X \subseteq \mathbb{S}^{n-1}$ *such that* $g(X) \geq \frac{c_2}{\sqrt{n}}\left(\frac{2}{\sqrt{3}}\right)^n$.

Proof:

1. Use Lemma 1 with $\varphi = \frac{\pi}{3}$ to get a thinly spread $X \subset \mathbb{S}^{n-1}$.
2. From (a), $\operatorname{diam} X \leq 2\sin(\pi/3) = \sqrt{3}$.
3. Any ball of diameter $\sqrt{3}$ intersects $\mathbb{S}^{n-1}$ by a cap of radius $\leq \pi/3$.
4. So by Lemma 3 (b) we need at least $\frac{c\sqrt{n}\log n}{(\sin\varphi)^n}/(Cn\log n) = \frac{c_2}{\sqrt{n}}\left(\frac{2}{\sqrt{3}}\right)^n$ such caps to cover X.

Theorem (A., Bondarenko, Prymak 2023)

For some constants c_1, c_2,

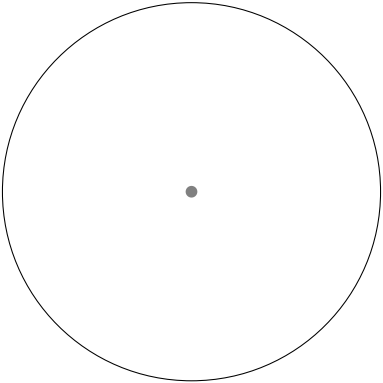
$$h(n) \geq \frac{c_1}{\sqrt{n}} \left(\frac{1}{\cos\left(\frac{\pi}{14}\right)} \right)^n \approx (1.0257...)^n,$$

$$g(n) \geq \frac{c_2}{\sqrt{n}} \left(\frac{2}{\sqrt{3}} \right)^n \approx (1.1547...)^n.$$

Illumination – more geometric definitions

Definition (Cone)

$Q(x,\varphi) = \{x\} \cup \{y \in \mathbb{S}^{n-1} : \theta(x,y) = \pi - 2\varphi\}$ for $x \in \mathbb{S}^{n-1}$, $\varphi \in [0,\pi/2]$.



$\varphi \leq \frac{\pi}{6} \Rightarrow \operatorname{diam}(Q(x,\varphi)) = 2\cos\varphi$.

Illumination Lemma

Fix small $\alpha \leq \frac{\pi}{6}$, and let $d = d(\alpha) = \mathrm{diam}\,(Q(x,\alpha)) = 2\cos\alpha$.

Lemma 2

Let $\mathrm{diam}\,(K) = d$ *and* $Q(x,\alpha) \subset K$*. If* $\xi \in \mathbb{S}^{n-1}$ *illuminates* x *in* K*, then* $x \in C(-\xi, \frac{\pi}{2} - \alpha)$*.*

Separation Lemma

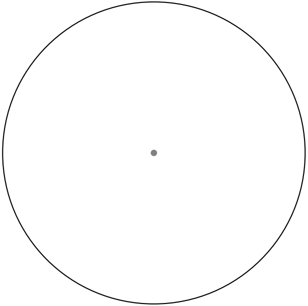
For $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$, let $W(X) = \bigcup_{x \in X} Q(x, \alpha)$.

Lemma 3

Let $X = \{x_1, \ldots, x_N\} \subset \mathbb{S}^{n-1}$ be such that for all distinct $x_i, x_j \in X$

$$4\alpha \leq \theta(x_i, x_j) \leq \pi - 6\alpha.$$

Then $\operatorname{diam}(W(X)) = 2\cos\alpha = diam(Q(x, \alpha))$.



Theorem (Illumination of constant width bodies)

There is a body of constant width $K \subset \mathbb{R}^n$ such that
$$I(K) \geq \frac{c_1}{\sqrt{n}} \left(\frac{1}{\cos(\pi/14)} \right)^n.$$

Theorem (Illumination of constant width bodies)

There is a body of constant width $K \subset \mathbb{R}^n$ such that
$$I(K) \geq \frac{c_1}{\sqrt{n}} \left(\frac{1}{\cos(\pi/14)} \right)^n.$$

Proof:

1. Use Lemma 1 with $\varphi = \frac{6\pi}{14}$ to get a thinly spread $X \subset \mathbb{S}^{n-1}$.

Theorem (Illumination of constant width bodies)

There is a body of constant width $K \subset \mathbb{R}^n$ such that
$I(K) \geq \frac{c_1}{\sqrt{n}} \left(\frac{1}{\cos(\pi/14)} \right)^n.$

Proof:

1. Use Lemma 1 with $\varphi = \frac{6\pi}{14}$ to get a thinly spread $X \subset \mathbb{S}^{n-1}$.
2. Let $W(X) = \bigcup_{x \in X} Q(x, \alpha)$ with $\alpha = \frac{\pi}{14}$. Recall $\operatorname{diam}(Q(x, \alpha)) = 2\cos(\alpha)$.

Theorem (Illumination of constant width bodies)

There is a body of constant width $K \subset \mathbb{R}^n$ such that
$$I(K) \geq \frac{c_1}{\sqrt{n}} \left(\frac{1}{\cos(\pi/14)} \right)^n .$$

Proof:

1. Use Lemma 1 with $\varphi = \frac{6\pi}{14}$ to get a thinly spread $X \subset \mathbb{S}^{n-1}$.
2. Let $W(X) = \bigcup_{x \in X} Q(x, \alpha)$ with $\alpha = \frac{\pi}{14}$. Recall $\operatorname{diam}(Q(x, \alpha)) = 2\cos(\alpha)$.
3. By Lemma 3 (separation lemma), $\operatorname{diam}(W(X)) = 2\cos\alpha$.

Theorem (Illumination of constant width bodies)

There is a body of constant width $K \subset \mathbb{R}^n$ such that
$$I(K) \geq \frac{c_1}{\sqrt{n}} \left(\frac{1}{\cos(\pi/14)} \right)^n.$$

Proof:

1. Use Lemma 1 with $\varphi = \frac{6\pi}{14}$ to get a thinly spread $X \subset \mathbb{S}^{n-1}$.
2. Let $W(X) = \bigcup_{x \in X} Q(x, \alpha)$ with $\alpha = \frac{\pi}{14}$. Recall $\operatorname{diam}(Q(x, \alpha)) = 2\cos(\alpha)$.
3. By Lemma 3 (separation lemma), $\operatorname{diam}(W(X)) = 2\cos\alpha$.
4. There is a body $K \supset W(X)$ of constant width $2\cos\alpha$.

Theorem (Illumination of constant width bodies)

There is a body of constant width $K \subset \mathbb{R}^n$ such that
$$I(K) \geq \frac{c_1}{\sqrt{n}} \left(\frac{1}{\cos(\pi/14)} \right)^n.$$

Proof:

1. Use Lemma 1 with $\varphi = \frac{6\pi}{14}$ to get a thinly spread $X \subset \mathbb{S}^{n-1}$.
2. Let $W(X) = \bigcup_{x \in X} Q(x, \alpha)$ with $\alpha = \frac{\pi}{14}$. Recall $\operatorname{diam}(Q(x, \alpha)) = 2\cos(\alpha)$.
3. By Lemma 3 (separation lemma), $\operatorname{diam}(W(X)) = 2\cos\alpha$.
4. There is a body $K \supset W(X)$ of constant width $2\cos\alpha$.
5. With $\varphi = \frac{\pi}{2} - \alpha$, Lemma 1 in combination with Lemma 2 (illumination lemma) imply $I(K) \geq \frac{c\sqrt{n}\log n}{(\sin\varphi)^n} / (Cn\log n) = \frac{c_1}{\sqrt{n}} \left(\frac{1}{\cos(\pi/14)} \right)^n$.

Theorem (Illumination of constant width bodies)

There is a body of constant width $K \subset \mathbb{R}^n$ such that
$I(K) \geq \frac{c_1}{\sqrt{n}} \left(\frac{1}{\cos(\pi/14)} \right)^n.$

Proof:

1. Use Lemma 1 with $\varphi = \frac{6\pi}{14}$ to get a thinly spread $X \subset \mathbb{S}^{n-1}$.
2. Let $W(X) = \bigcup_{x \in X} Q(x, \alpha)$ with $\alpha = \frac{\pi}{14}$. Recall $\operatorname{diam}(Q(x, \alpha)) = 2\cos(\alpha)$.
3. By Lemma 3 (separation lemma), $\operatorname{diam}(W(X)) = 2\cos\alpha$.
4. There is a body $K \supset W(X)$ of constant width $2\cos\alpha$.
5. With $\varphi = \frac{\pi}{2} - \alpha$, Lemma 1 in combination with Lemma 2 (illumination lemma) imply
$I(K) \geq \frac{c\sqrt{n}\log n}{(\sin\varphi)^n} / (Cn \log n) = \frac{c_1}{\sqrt{n}} \left(\frac{1}{\cos(\pi/14)} \right)^n.$

Glazyrin (2023) noted that the base of the exponent $\frac{1}{\cos(\pi/14)} \approx 1.026$ can be improved to $\frac{1}{4}\sqrt{\frac{1}{6}(111 - \sqrt{33})} \approx 1.047$ by modification of the construction: choosing the bases of the cones from a concentric sphere of smaller radius.

More questions?

More questions?

Our constructions of bodies of constant width also provide the same exponential lower bounds for “mix and match” covering by balls of smaller diameter and smaller homothets.

More questions?

Our constructions of bodies of constant width also provide the same exponential lower bounds for "mix and match" covering by balls of smaller diameter and smaller homothets.

Question

Can $b(n) \leq (\sqrt{3/2} + o(1))^n$ *be improved using "mix and match" covering by balls of smaller diameters and smaller homothets?*

More questions?

Our constructions of bodies of constant width also provide the same exponential lower bounds for "mix and match" covering by balls of smaller diameter and smaller homothets.

Question

Can $b(n) \leq (\sqrt{3/2} + o(1))^n$ *be improved using "mix and match" covering by balls of smaller diameters and smaller homothets?*

Question

Is it true that $I(K) = g(K)$ *for any* K *of constant width? If not, are* $I(K)$ *and* $g(K)$ *equivalent up to a polynomial in* n*?*

Lebesgue's universal covering problem

A measurable set $U \subset \mathbb{E}^n$ is called a **universal cover** if every set $A \subset \mathbb{E}^n$ of diameter 1 is contained in a congruent copy of U.

Lebesgue's universal covering problem

A measurable set $U \subset \mathbb{E}^n$ is called a **universal cover** if every set $A \subset \mathbb{E}^n$ of diameter 1 is contained in a congruent copy of U.

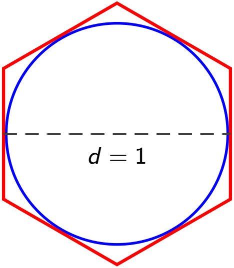
1914: Lebesgue asked Pál in a letter what is the smallest area A of a convex universal cover in the plane ($n = 2$).

Lebesgue's universal covering problem

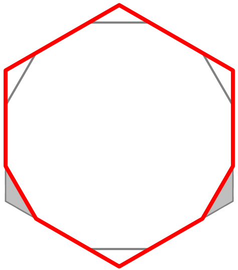
A measurable set $U \subset \mathbb{E}^n$ is called a **universal cover** if every set $A \subset \mathbb{E}^n$ of diameter 1 is contained in a congruent copy of U.

1914: Lebesgue asked Pál in a letter what is the smallest area A of a convex universal cover in the plane ($n = 2$).

1920: Pál showed that a regular hexagon circumscribed about a circle of diameter 1 is a universal cover, so $A \leq \frac{\sqrt{3}}{2} \approx 0.8660254$.

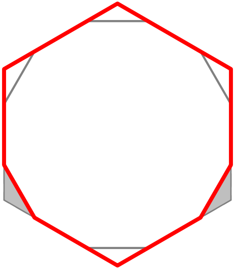


1920: Pál further considered largest regular dodecagon in that hexagon, and observed that two triangles can be removed while keeping the universal cover property, yielding $A \leq 2 - \frac{2}{\sqrt{3}} \approx 0.84529946$.



Lebesgue's universal covering problem

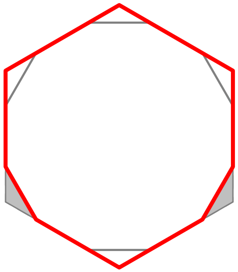
1920: Pál further considered largest regular dodecagon in that hexagon, and observed that two triangles can be removed while keeping the universal cover property, yielding $A \leq 2 - \frac{2}{\sqrt{3}} \approx 0.84529946$.



1936: $A \leq 0.844137708436$ by Sprague.

Lebesgue's universal covering problem

1920: Pál further considered largest regular dodecagon in that hexagon, and observed that two triangles can be removed while keeping the universal cover property, yielding $A \leq 2 - \frac{2}{\sqrt{3}} \approx 0.84529946$.



1936: $A \leq 0.844137708436$ by Sprague.
1992: $A \leq 0.844137708398$ by Hansen.

Lebesgue's universal covering problem

1992: $A \leq 0.844137708398$ by Hansen.

The problem of determining A is still not solved. Records:

Lebesgue's universal covering problem

1992: $A \leq 0.844137708398$ by Hansen.

The problem of determining A is still not solved. Records:

2005: $0.832 \leq A$ by Brass and Sharifi.

Lebesgue's universal covering problem

1992: $A \leq 0.844137708398$ by Hansen.

The problem of determining A is still not solved. Records:

2005: $0.832 \leq A$ by Brass and Sharifi.

2015: $A \leq 0.8441153$ by Baez, Bagdasaryan and Gibbs.

1992: $A \leq 0.844137708398$ by Hansen.

The problem of determining A is still not solved. Records:

2005: $0.832 \leq A$ by Brass and Sharifi.

2015: $A \leq 0.8441153$ by Baez, Bagdasaryan and Gibbs.

2018: $A \leq 0.8440935944$ by Gibbs (pre-print).

Asymptotic Lebesgue's universal covering problem

B_n — the unit ball in $\mathbb{E}^n$
$\mathrm{Vol}(\cdot)$ — the Lebesgue measure in $\mathbb{E}^n$

Asymptotic Lebesgue's universal covering problem

B_n — the unit ball in $\mathbb{E}^n$
$\mathrm{Vol}(\cdot)$ — the Lebesgue measure in $\mathbb{E}^n$

Jung's theorem: $J_n := r_n B_n$, where $r_n := \sqrt{\frac{n}{2n+2}}$, is a universal cover in $\mathbb{E}^n$.

Asymptotic Lebesgue's universal covering problem

B_n — the unit ball in $\mathbb{E}^n$
$\mathrm{Vol}(\cdot)$ — the Lebesgue measure in $\mathbb{E}^n$

Jung's theorem: $J_n := r_n B_n$, where $r_n := \sqrt{\frac{n}{2n+2}}$, is a universal cover in $\mathbb{E}^n$.

Our main result: any universal cover has volume at least $(1-o(1))^n \mathrm{Vol}(J_n)$, so J_n **is an asymptotically optimal universal cover**.

Asymptotic Lebesgue's universal covering problem

B_n — the unit ball in $\mathbb{E}^n$
$\mathrm{Vol}(\cdot)$ — the Lebesgue measure in $\mathbb{E}^n$

Jung's theorem: $J_n := r_n B_n$, where $r_n := \sqrt{\frac{n}{2n+2}}$, is a universal cover in $\mathbb{E}^n$.

Our main result: any universal cover has volume at least $(1-o(1))^n \mathrm{Vol}(J_n)$, so J_n **is an asymptotically optimal universal cover**.

Theorem

Let U be a universal cover in $\mathbb{E}^n$. Then

$$\mathrm{Vol}(U) \geq \exp\left(-\sqrt{\left(\tfrac{5}{4} + o(1)\right) n \log n}\right) \mathrm{Vol}(J_n)\,.$$

Asymptotic Lebesgue's universal covering problem

B_n — the unit ball in $\mathbb{E}^n$
$\mathrm{Vol}(\cdot)$ — the Lebesgue measure in $\mathbb{E}^n$

Jung's theorem: $J_n := r_n B_n$, where $r_n := \sqrt{\frac{n}{2n+2}}$, is a universal cover in $\mathbb{E}^n$.

Our main result: any universal cover has volume at least $(1-o(1))^n \mathrm{Vol}(J_n)$, so J_n **is an asymptotically optimal universal cover**.

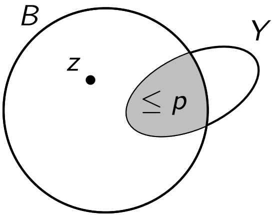
Theorem

Let U be a universal cover in $\mathbb{E}^n$. Then

$$\mathrm{Vol}(U) \geq \exp\left(-\sqrt{\left(\tfrac{5}{4}+o(1)\right) n \log n}\right) \mathrm{Vol}(J_n)\,.$$

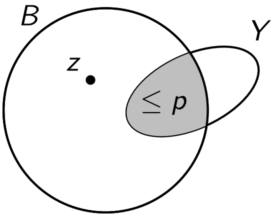
Path of the proof: assuming a minimal volume universal cover U is too small, we construct a set of diameter < 1 which cannot be covered by any congruent copy of U.

Proof: avoiding a single small set



$\nu(B) = 1$, $\nu(Y \cap B) \leq p$, $0 < p < 1$.

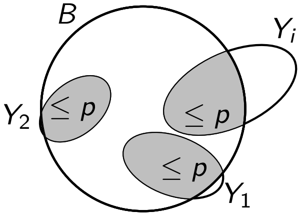
Proof: avoiding a single small set



$\nu(B) = 1$, $\nu(Y \cap B) \leq p$, $0 < p < 1$.

A ν-random single point from B avoids Y with prob. $\geq 1 - p$.

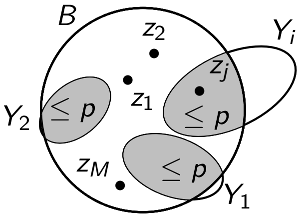
Proof: avoiding sets from a collection



$\nu(B) = 1$, $0 < p < 1$, $\mathcal{Y} = \{Y_i\}$.

$\nu(Y_i \cap B) \leq p$ for each $Y_i \in \mathcal{Y}$.

Proof: avoiding sets from a collection

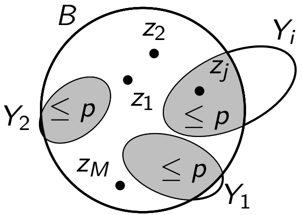


$\nu(B) = 1$, $0 < p < 1$, $\mathcal{Y} = \{Y_i\}$.

$\nu(Y_i \cap B) \leq p$ for each $Y_i \in \mathcal{Y}$.

Take M i.i.d. points in B w.r.t. ν:
$Z := \{z_1, \ldots z_M\}$.

Proof: avoiding sets from a collection



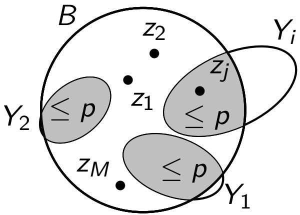
$\nu(B) = 1$, $0 < p < 1$, $\mathcal{Y} = \{Y_i\}$.

$\nu(Y_i \cap B) \leq p$ for each $Y_i \in \mathcal{Y}$.

Take M i.i.d. points in B w.r.t. ν:
$Z := \{z_1, \ldots z_M\}$.

Then $|Z \cap Y_i| \sim \mathrm{Binomial}(M, \tilde{p})$, where
$\tilde{p} = \nu(Y_i \cap B) \leq p$.

Proof: avoiding sets from a collection



$\nu(B) = 1$, $0 < p < 1$, $\mathcal{Y} = \{Y_i\}$.

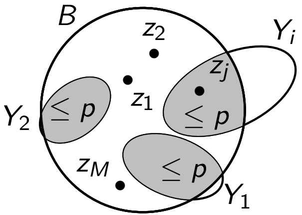
$\nu(Y_i \cap B) \leq p$ for each $Y_i \in \mathcal{Y}$.

Take M i.i.d. points in B w.r.t. ν:
$Z := \{z_1, \ldots z_M\}$.

Then $|Z \cap Y_i| \sim \mathrm{Binomial}(M, \tilde{p})$, where
$\tilde{p} = \nu(Y_i \cap B) \leq p$.

By Chernoff's bound:
$\mathbb{P}(|Z \cap Y_i| \geq \frac{M}{2}) < (2ep)^{\frac{M}{2}}$.

Proof: avoiding sets from a collection



$\nu(B) = 1$, $0 < p < 1$, $\mathcal{Y} = \{Y_i\}$.

$\nu(Y_i \cap B) \leq p$ for each $Y_i \in \mathcal{Y}$.

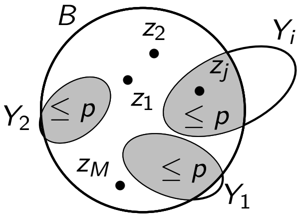
Take M i.i.d. points in B w.r.t. ν:
$Z := \{z_1, \ldots z_M\}$.

Then $|Z \cap Y_i| \sim \mathrm{Binomial}(M, \tilde{p})$, where
$\tilde{p} = \nu(Y_i \cap B) \leq p$.

By Chernoff's bound:
$\mathbb{P}(|Z \cap Y_i| \geq \frac{M}{2}) < (2ep)^{\frac{M}{2}}$.

By the union bound, if $|\mathcal{Y}|(2ep)^{\frac{M}{2}} < \frac{1}{2}$, then with prob. $\geq \frac{1}{2}$
$|Z \cap Y_i| < \frac{M}{2}$ for any Y_i.

Proof: avoiding sets from a collection



$\nu(B) = 1$, $0 < p < 1$, $\mathcal{Y} = \{Y_i\}$.

$\nu(Y_i \cap B) \leq p$ for each $Y_i \in \mathcal{Y}$.

Take M i.i.d. points in B w.r.t. ν:
$Z := \{z_1, \dots z_M\}$.

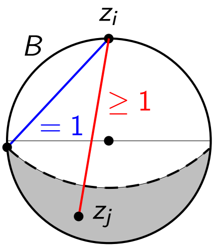
Then $|Z \cap Y_i| \sim \mathrm{Binomial}(M, \tilde{p})$, where
$\tilde{p} = \nu(Y_i \cap B) \leq p$.

By Chernoff's bound:
$\mathbb{P}(|Z \cap Y_i| \geq \frac{M}{2}) < (2ep)^{\frac{M}{2}}$.

By the union bound, if $|\mathcal{Y}|(2ep)^{\frac{M}{2}} < \frac{1}{2}$, then with prob. $\geq \frac{1}{2}$
$|Z \cap Y_i| < \frac{M}{2}$ for any Y_i.

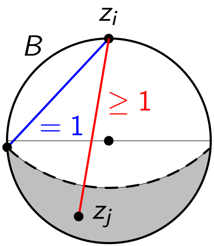
In other words, if $|\mathcal{Y}|(2ep)^{\frac{M}{2}} < \frac{1}{2}$, then with prob. $\geq \frac{1}{2}$
any Y_i covers less than half of Z.

Proof: controlling diameter



B is close to J_n, ν is uniform on B.

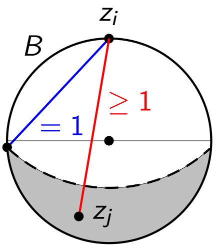
Proof: controlling diameter



B is close to J_n, ν is uniform on B.

We need $\mathbb{P}(\|z_i - z_j\| \geq 1) < \frac{1}{2M}$.

Proof: controlling diameter

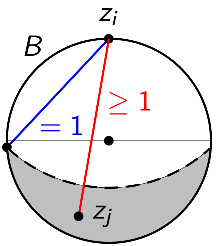


B is close to J_n, ν is uniform on B.

We need $\mathbb{P}(\|z_i - z_j\| \geq 1) < \frac{1}{2M}$.

Among $\binom{M}{2}$ pairs (z_i, z_j) the expected number of "bad" (far) pairs is $< \binom{M}{2}\frac{1}{2M} < \frac{M}{4}$,

Proof: controlling diameter

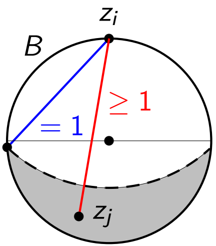


B is close to J_n, ν is uniform on B.

We need $\mathbb{P}(\|z_i - z_j\| \geq 1) < \frac{1}{2M}$.

Among $\binom{M}{2}$ pairs (z_i, z_j) the expected number of "bad" (far) pairs is $< \binom{M}{2}\frac{1}{2M} < \frac{M}{4}$, so by Markov's inequality, with prob. $> \frac{1}{2}$ there is $< \frac{M}{2}$ bad pairs.

Proof: controlling diameter

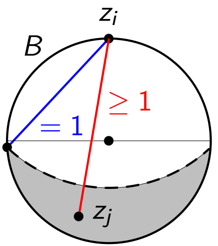


B is close to J_n, ν is uniform on B.

We need $\mathbb{P}(\|z_i - z_j\| \geq 1) < \frac{1}{2M}$.

Among $\binom{M}{2}$ pairs (z_i, z_j) the expected number of "bad" (far) pairs is $< \binom{M}{2}\frac{1}{2M} < \frac{M}{4}$, so by Markov's inequality, with prob. $> \frac{1}{2}$ there is $< \frac{M}{2}$ bad pairs. Removing a point from each pair, we get $X \subset Z$ with $|X| \geq \frac{M}{2}$ and $\operatorname{diam}(X) < 1$ while $|X \cap Y_i| < \frac{M}{2}$ for each $Y_i \in \mathcal{Y}$.

Proof: controlling diameter



B is close to J_n, ν is uniform on B.

We need $\mathbb{P}(\|z_i - z_j\| \geq 1) < \frac{1}{2M}$.

Among $\binom{M}{2}$ pairs (z_i, z_j) the expected number of "bad" (far) pairs is $< \binom{M}{2}\frac{1}{2M} < \frac{M}{4}$, so by Markov's inequality, with prob. $> \frac{1}{2}$ there is $< \frac{M}{2}$ bad pairs. Removing a point from each pair, we get $X \subset Z$ with $|X| \geq \frac{M}{2}$ and $\operatorname{diam}(X) < 1$ while $|X \cap Y_i| < \frac{M}{2}$ for each $Y_i \in \mathcal{Y}$.

Thus X cannot be covered by any $Y_i \in \mathcal{Y}$ and $\operatorname{diam}(X) < 1$.

Lower bound on covering in terms of measurable graphs

A graph Γ is a pair (V, E) where $E \subseteq V \times V$ is the set of edges of Γ.
We assume that Γ is unoriented: $(x, y) \in E \iff (y, x) \in E$.
Let $\Delta(V)$ be the diagonal $\{(v, v) \colon v \in V\} \subseteq V \times V$.

Lower bound on covering in terms of measurable graphs

A graph Γ is a pair (V, E) where $E \subseteq V \times V$ is the set of edges of Γ.
We assume that Γ is unoriented: $(x, y) \in E \iff (y, x) \in E$.
Let $\Delta(V)$ be the diagonal $\{(v, v) \colon v \in V\} \subseteq V \times V$.

Given a set Ω, a family $\mathcal{Y}$ of subsets of Ω, and $X \subset \Omega$, define

$$N(X, \mathcal{Y}) := \min\left\{|\mathcal{U}| : X \subset \bigcup_{Y \in \mathcal{U}} Y, \mathcal{U} \subset \mathcal{Y}\right\},$$

i.e. $N(X, \mathcal{Y})$ is the minimal number of sets from $\mathcal{Y}$ needed to cover X.

Lower bound on covering in terms of measurable graphs

Theorem

Let $\Gamma = (V, E)$ be a graph and assume that V is equipped with a probability measure ν satisfying $(\nu \times \nu)(\Delta(V)) = 0$.
Let $\mathcal{Y}$ be a finite family of measurable subsets of V.
Assume $0 < p < \frac{1}{2}$, $M, k \in \mathbb{Z}_{>0}$ and $1 \leq k \leq \frac{1}{2p}$. If

$$\nu(Y) \leq p \quad \text{for all} \quad Y \in \mathcal{Y}, \quad \text{(each covering set is “small”)}$$

$$|\mathcal{Y}| \leq \frac{1}{2}(2ekp)^{-\frac{M}{2k}}, \quad \text{(number of covering sets is not “huge”)} \quad \text{and}$$

$$(\nu \times \nu)(E) \leq \frac{1}{2M}, \quad \text{(not many “undesirable” pairs)}$$

Lower bound on covering in terms of measurable graphs

Theorem

Let $\Gamma = (V, E)$ be a graph and assume that V is equipped with a probability measure ν satisfying $(\nu \times \nu)(\Delta(V)) = 0$.
Let $\mathcal{Y}$ be a finite family of measurable subsets of V.
Assume $0 < p < \frac{1}{2}$, $M, k \in \mathbb{Z}_{>0}$ and $1 \le k \le \frac{1}{2p}$. If

$$\nu(Y) \le p \quad \textit{for all} \quad Y \in \mathcal{Y}, \qquad \textit{(each covering set is "small")}$$

$$|\mathcal{Y}| \le \frac{1}{2}(2ekp)^{-\frac{M}{2k}}, \qquad \textit{(number of covering sets is not "huge")} \quad \textit{and}$$

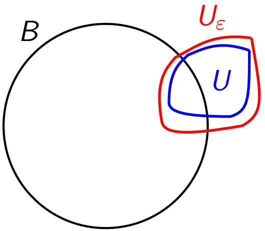
$$(\nu \times \nu)(E) \le \frac{1}{2M}, \qquad \textit{(not many "undesirable" pairs)}$$

then there exists a coclique $X \subset V$ such that

$$|X| \ge M/2 \quad \textit{and} \quad N(X, \mathcal{Y}) > k\,.$$

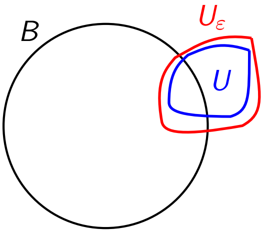
(Large hard-to-cover set without undesirable pairs.)

Proof: ε-thickening and discretization of covering family



For a minimal universal cover U, the family $\mathcal{I}_U = \{T(U) : T(U) \cap B \neq \emptyset, T \text{ isometry}\}$ is infinite, so not suitable directly.

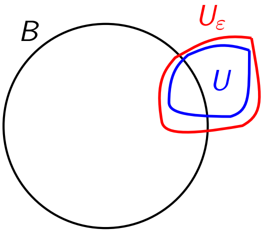
Proof: ε-thickening and discretization of covering family



For a minimal universal cover U, the family $\mathcal{I}_U = \{T(U) : T(U) \cap B \neq \emptyset, T \text{ isometry}\}$ is infinite, so not suitable directly.

With proper $\varepsilon > 0$, we take $U_\varepsilon := U + \varepsilon B_n$ and show $\mathrm{Vol}(U_\varepsilon) < 2\mathrm{Vol}(U)$, and

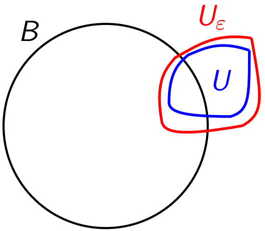
Proof: ε-thickening and discretization of covering family



For a minimal universal cover U, the family $\mathcal{I}_U = \{T(U) : T(U) \cap B \neq \emptyset, T \text{ isometry}\}$ is infinite, so not suitable directly.

With proper $\varepsilon > 0$, we take $U_\varepsilon := U + \varepsilon B_n$ and show $\mathrm{Vol}(U_\varepsilon) < 2\mathrm{Vol}(U)$, and there exists $\mathcal{Y} \subset \mathcal{I}_{U_\varepsilon}$, $|\mathcal{Y}| < \frac{1}{2} n^{n^3}$, such that any set from $\mathcal{I}_U$ is a subset of some $Y \in \mathcal{Y}$.

Proof: ε-thickening and discretization of covering family



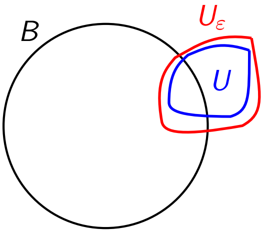
For a minimal universal cover U, the family $\mathcal{I}_U = \{T(U) : T(U) \cap B \neq \emptyset, T \text{ isometry}\}$ is infinite, so not suitable directly.

With proper $\varepsilon > 0$, we take $U_\varepsilon := U + \varepsilon B_n$ and show $\mathrm{Vol}(U_\varepsilon) < 2\mathrm{Vol}(U)$, and there exists $\mathcal{Y} \subset \mathcal{I}_{U_\varepsilon}$, $|\mathcal{Y}| < \frac{1}{2} n^{n^3}$, such that any set from $\mathcal{I}_U$ is a subset of some $Y \in \mathcal{Y}$.

Some ingredients used:

- $\mathrm{diam}\,(U) < n^{n/2}$ due to Makeev, 1990;
- a bound on size of ε-net in orthogonal group by Szarek, 1981.

Proof: ε-thickening and discretization of covering family



For a minimal universal cover U, the family $\mathcal{I}_U = \{T(U) : T(U) \cap B \neq \emptyset, T \text{ isometry}\}$ is infinite, so not suitable directly.

With proper $\varepsilon > 0$, we take $U_\varepsilon := U + \varepsilon B_n$ and show $\mathrm{Vol}(U_\varepsilon) < 2\mathrm{Vol}(U)$, and there exists $\mathcal{Y} \subset \mathcal{I}_{U_\varepsilon}$, $|\mathcal{Y}| < \frac{1}{2} n^{n^3}$, such that any set from $\mathcal{I}_U$ is a subset of some $Y \in \mathcal{Y}$.

Some ingredients used:

- $\mathrm{diam}\,(U) < n^{n/2}$ due to Makeev, 1990;
- a bound on size of ε-net in orthogonal group by Szarek, 1981.

Any subset of B of diameter ≤ 1 is covered by an element of $\mathcal{Y}$, $|\mathcal{Y}| < \frac{1}{2} n^{n^3}$.

Conclusion of the proof

Theorem

Let U be a universal cover in $\mathbb{E}^n$. Then

$$\mathrm{Vol}(U) \geq \exp\left(-\sqrt{(\tfrac{5}{4} + o(1))\, n \log n}\right) \mathrm{Vol}(J_n)\,.$$

If $\mathrm{Vol}(U)$ does not satisfy the estimate, with suitable choices of the radius of B and ε, p, M, the above probabilistic framework constructs $X \subset B$ of diameter ≤ 1 which cannot be covered by a congruent copy of U, a contradiction.

Universal covers for estimates on Borsuk's number

For small n, partitioning a universal cover into pieces of diameter < 1 can give a good upper bound on $b(n)$.

Universal covers for estimates on Borsuk's number

For small n, partitioning a universal cover into pieces of diameter < 1 can give a good upper bound on $b(n)$.

1920: Pál showed the regular hexagon circumscribed about unit disc is a universal cover, implying $b(2) \leq 3$.

Universal covers for estimates on Borsuk's number

For small n, partitioning a universal cover into pieces of diameter < 1 can give a good upper bound on $b(n)$.

1920: Pál showed the regular hexagon circumscribed about unit disc is a universal cover, implying $b(2) \leq 3$.

1955: Eggleston; 1957: Grünbaum used suitably truncated octahedra to obtain $b(3) \leq 4$.

Universal covers for estimates on Borsuk's number

For small n, partitioning a universal cover into pieces of diameter < 1 can give a good upper bound on $b(n)$.

1920: Pál showed the regular hexagon circumscribed about unit disc is a universal cover, implying $b(2) \leq 3$.

1955: Eggleston; 1957: Grünbaum used suitably truncated octahedra to obtain $b(3) \leq 4$.

1982: Lassak observed $J_n \cap (r_n u + B_n)$, $\|u\| = 1$, is a universal cover, and then proved $b(n) \leq 2^{n-1} + 1$, which remains best known for $4 \leq n \leq 17$.

Universal covers for estimates on Borsuk's number

For small n, partitioning a universal cover into pieces of diameter < 1 can give a good upper bound on $b(n)$.

1920: Pál showed the regular hexagon circumscribed about unit disc is a universal cover, implying $b(2) \le 3$.

1955: Eggleston; 1957: Grünbaum used suitably truncated octahedra to obtain $b(3) \le 4$.

1982: Lassak observed $J_n \cap (r_n u + B_n)$, $\|u\| = 1$, is a universal cover, and then proved $b(n) \le 2^{n-1} + 1$, which remains best known for $4 \le n \le 17$.

Corollary

If a universal cover in $\mathbb{E}^n$ is partitioned into M pieces of diameter ≤ 1, then $M \ge (\sqrt{2} - o(1))^n$.

Universal covers for estimates on Borsuk's number

For small n, partitioning a universal cover into pieces of diameter < 1 can give a good upper bound on $b(n)$.

1920: Pál showed the regular hexagon circumscribed about unit disc is a universal cover, implying $b(2) \leq 3$.

1955: Eggleston; 1957: Grünbaum used suitably truncated octahedra to obtain $b(3) \leq 4$.

1982: Lassak observed $J_n \cap (r_n u + B_n)$, $\|u\| = 1$, is a universal cover, and then proved $b(n) \leq 2^{n-1} + 1$, which remains best known for $4 \leq n \leq 17$.

Corollary

If a universal cover in $\mathbb{E}^n$ is partitioned into M pieces of diameter ≤ 1, then $M \geq (\sqrt{2} - o(1))^n$.

Proof: isodiametric inequality and simple volume estimates.

Universal covers for estimates on Borsuk's number

For small n, partitioning a universal cover into pieces of diameter < 1 can give a good upper bound on $b(n)$.

1920: Pál showed the regular hexagon circumscribed about unit disc is a universal cover, implying $b(2) \leq 3$.

1955: Eggleston; 1957: Grünbaum used suitably truncated octahedra to obtain $b(3) \leq 4$.

1982: Lassak observed $J_n \cap (r_n u + B_n)$, $\|u\| = 1$, is a universal cover, and then proved $b(n) \leq 2^{n-1} + 1$, which remains best known for $4 \leq n \leq 17$.

Corollary

If a universal cover in $\mathbb{E}^n$ is partitioned into M pieces of diameter ≤ 1, then $M \geq (\sqrt{2} - o(1))^n$.

Proof: isodiametric inequality and simple volume estimates.

Recall that $b(n) \leq \left(\sqrt{\frac{3}{2}} + o(1)\right)^n$ Schramm; Bourgain and Lindenstrauss.

Asymptotic optimality of J_n for other measures

Mean width $w(K)$ of a convex body K in $\mathbb{E}^n$ is half the average length of a projection of K onto a line, e.g. $w(B_n) = 1$.

Asymptotic optimality of J_n for other measures

Mean width $w(K)$ of a convex body K in $\mathbb{E}^n$ is half the average length of a projection of K onto a line, e.g. $w(B_n) = 1$.

1990 Makeev; 1998 Bezdek and Connelly:
J_n minimizes **mean width** of **translative** universal covers.

Asymptotic optimality of J_n for other measures

Mean width $w(K)$ of a convex body K in $\mathbb{E}^n$ is half the average length of a projection of K onto a line, e.g. $w(B_n) = 1$.

1990 Makeev; 1998 Bezdek and Connelly:
J_n minimizes **mean width** of **translative** universal covers.

For the general congruent (non-translative) universal covers J_n is not a minimizer of mean width or volume: $J_n \cap (r_n u + B_n)$, $\|u\| = 1$, is smaller.

Asymptotic optimality of J_n for other measures

Mean width $w(K)$ of a convex body K in $\mathbb{E}^n$ is half the average length of a projection of K onto a line, e.g. $w(B_n) = 1$.

1990 Makeev; 1998 Bezdek and Connelly:
J_n minimizes **mean width** of **translative** universal covers.

For the general congruent (non-translative) universal covers J_n is not a minimizer of mean width or volume: $J_n \cap (r_n u + B_n)$, $\|u\| = 1$, is smaller.

Corollary

$w(U) \geq (1 - o(1))w(J_n)$ for any convex universal cover U.

Asymptotic optimality of J_n for other measures

Mean width $w(K)$ of a convex body K in $\mathbb{E}^n$ is half the average length of a projection of K onto a line, e.g. $w(B_n) = 1$.

1990 Makeev; 1998 Bezdek and Connelly:
J_n minimizes **mean width** of **translative** universal covers.

For the general congruent (non-translative) universal covers J_n is not a minimizer of mean width or volume: $J_n \cap (r_n u + B_n)$, $\|u\| = 1$, is smaller.

Corollary

$w(U) \geq (1 - o(1)) w(J_n)$ for any convex universal cover U.

Proof: Urysohn's inequality $w(U) \geq \left(\frac{\mathrm{Vol}(U)}{\mathrm{Vol}(B_n)}\right)^{\frac{1}{n}}$.

Asymptotic optimality of J_n for other measures

Mean width $w(K)$ of a convex body K in $\mathbb{E}^n$ is half the average length of a projection of K onto a line, e.g. $w(B_n) = 1$.

1990 Makeev; 1998 Bezdek and Connelly:
J_n minimizes **mean width** of **translative** universal covers.

For the general congruent (non-translative) universal covers J_n is not a minimizer of mean width or volume: $J_n \cap (r_n u + B_n)$, $\|u\| = 1$, is smaller.

Corollary

$w(U) \geq (1 - o(1))w(J_n)$ for any convex universal cover U.

Proof: Urysohn's inequality $w(U) \geq \left(\frac{\mathrm{Vol}(U)}{\mathrm{Vol}(B_n)}\right)^{\frac{1}{n}}$.

Remark

Using Alexandrov inequality, the corollary can be extended to all quermassintegrals/intrinsic volumes, e.g. to the surface area.

Application to quermassintegrals

By Steiner's formula, for any convex body K in $\mathbb{E}^n$

$$\mathrm{Vol}(K + tB_n) = \sum_{j=0}^{n} \binom{n}{j} W_j(K) t^j, \quad t \geq 0,$$

where $W_j(K)$ is the j-th quermassintegral. In particular, $W_0(K) = \mathrm{Vol}(K)$, $W_1(K) = \frac{1}{n}\partial(K)$, where $\partial(K)$ is the surface area of K, $W_{n-1}(K) = \frac{\mathrm{Vol}(B_n)}{2} w(K)$, and $W_n(K) = \mathrm{Vol}(B_n)$.

Application to quermassintegrals

By Steiner's formula, for any convex body K in $\mathbb{E}^n$

$$\mathrm{Vol}(K + tB_n) = \sum_{j=0}^{n} \binom{n}{j} W_j(K) t^j, \quad t \geq 0,$$

where $W_j(K)$ is the j-th quermassintegral. In particular, $W_0(K) = \mathrm{Vol}(K)$, $W_1(K) = \frac{1}{n}\partial(K)$, where $\partial(K)$ is the surface area of K, $W_{n-1}(K) = \frac{\mathrm{Vol}(B_n)}{2} w(K)$, and $W_n(K) = \mathrm{Vol}(B_n)$.
Alexandrov's inequality gives log-concavity of the sequence $\{W_j(K)\}_{j=0}^{n}$, yielding $W_k(K) \geq \mathrm{Vol}(K)^{\frac{n-k}{n}} \mathrm{Vol}(B_n)^{\frac{k}{n}}$.

Application to quermassintegrals

By Steiner's formula, for any convex body K in $\mathbb{E}^n$

$$\mathrm{Vol}(K + tB_n) = \sum_{j=0}^{n} \binom{n}{j} W_j(K) t^j, \quad t \geq 0,$$

where $W_j(K)$ is the j-th quermassintegral. In particular, $W_0(K) = \mathrm{Vol}(K)$, $W_1(K) = \frac{1}{n}\partial(K)$, where $\partial(K)$ is the surface area of K, $W_{n-1}(K) = \frac{\mathrm{Vol}(B_n)}{2} w(K)$, and $W_n(K) = \mathrm{Vol}(B_n)$.
Alexandrov's inequality gives log-concavity of the sequence $\{W_j(K)\}_{j=0}^{n}$, yielding $W_k(K) \geq \mathrm{Vol}(K)^{\frac{n-k}{n}} \mathrm{Vol}(B_n)^{\frac{k}{n}}$.

Corollary

Let U be a convex universal cover in $\mathbb{E}^n$. Then

$$W_k(U) \geq \exp\left(-(1 - \tfrac{k}{n}) \sqrt{(\tfrac{5}{4} + o(1))\, n \log n} \right) W_k(J_n).$$

More details at:

- https://arxiv.org/abs/2512.04023.
- https://arxiv.org/abs/2304.10418.

More details at:

- `https://arxiv.org/abs/2512.04023`.
- `https://arxiv.org/abs/2304.10418`.

Question

Do there exist universal covers U with $\mathrm{Vol}(U) \leq 0.999\mathrm{Vol}(J_n)$?

More details at:

- https://arxiv.org/abs/2512.04023.
- https://arxiv.org/abs/2304.10418.

Question

Do there exist universal covers U with $\mathrm{Vol}(U) \leq 0.999\mathrm{Vol}(J_n)$?

Thank you!