

AN IMPROVED UPPER BOUND FOR BORSUK’S PARTITION QUESTION

ABSTRACT. We prove $\limsup_{n \rightarrow \infty} b(n)^{1/n} < 1.166374$ for the Borsuk partition number. Two new ingredients strengthen a Gaussian common-cone transport. First, its first and third Hermite Jacobian components share an operator variance budget. Full tensor symmetry turns this restriction into a nonlinear scalar constraint and a spectral refinement. Second, we follow Jacobian moments through the Ornstein–Uhlenbeck smoothing trajectory and retain a commutator term in the fourth-moment flow. The resulting degree-specific entropy bound combines with the Hessian constraints in an exact rational certificate. A short version, using small rational parameters and $(x - 1)^4 \geq 0$, already gives a base below 1.1673. We also obtain a quantitative improvement for original acute cones with a statistical-dimension deficit, and justify selection among correlated Gaussian proposals at no leading entropy cost. We explain why the latter refinement is inactive at the current relaxed worst profile and distinguish these limitations from lower bounds for Borsuk’s number.

1. INTRODUCTION

Let $b(n)$ denote the least number of sets of strictly smaller diameter needed to partition every bounded set of positive diameter in $\mathbb{R}^n$. Borsuk’s question and the counterexample of Kahn and Kalai initiated a substantial study of this quantity [1, 2, 10]. Schramm’s illumination argument and the ball-covering argument of Bourgain and Lindenstrauss give the classical exponential base $\sqrt{3/2}$ [3, 4]. The elementary note [11] improves this base to $\sqrt{9536/6375} = 1.22304666\dots$ using cone reflection and a Gaussian shift. The present note develops a stronger argument: vertices in each spherical part share one self-dual cone, a smoothed reflection changes the Gaussian proposal, and relative entropy controls its cost. Matrix variance budgets and the evolution of noncommuting Jacobian moments through Gaussian smoothing further sharpen the estimate. All reductions and analytic ingredients specific to the present bound are included below.

Theorem 1.1. *The Borsuk partition number satisfies*

$$\limsup_{n \rightarrow \infty} b(n)^{1/n} \leq C_* = 1.166373221598\dots < 1.166374,$$

where (53) defines C_* exactly.

We first isolate a simpler estimate, with a short rational certificate.

Date: October 6, 2026 (2610-02v1).

Origin and curation. The main proof arguments were primarily generated using *ChatGPT-6 Astra*, Sept.–Oct. 2026 with discussion and prompts by Andrii Arman, Andriy Bondarenko, Andriy Prymak and Danylo Radchenko. Andriy Prymak curates this note but does not claim authorship of its mathematical results.

Mathematical context. A further improvement of the exponential upper bound for Borsuk’s partition number, following the elementary note [11]. The argument combines a common self-dual cone, Gaussian transport, and matrix constraints on Hermite coefficients. The broader context includes work on spherical bodies of constant width by Hall, Prymak and Sujsuntinukul [12]. Sources of the proof ingredients are credited in the text.

Verification: not verified. The mathematical claims and proofs in this note have not been verified by the curator. The accompanying Python program checks specific numerical certificates; it does not verify the analytic arguments.

Permanent URL. <https://prymak.net/aiomn/2610-02v1.pdf>.

License. To the extent that Andriy Prymak holds copyright or related rights in this manuscript and its L^AT_EX source, those rights are licensed under [Creative Commons Attribution 4.0 International \(CC BY 4.0\)](https://creativecommons.org/licenses/by/4.0/). Third-party material retains its own terms.

Cite as. *An improved upper bound for Borsuk’s partition question.* AI-originated mathematical notes, curated by Andriy Prymak, version 2610-02v1, October 6, 2026. <https://prymak.net/aiomn/2610-02v1.pdf>. Identify the curator’s role without assigning mathematical authorship.

Theorem 1.2. *One has*

$$\limsup_{n \rightarrow \infty} b(n)^{1/n} \leq \sqrt{\frac{853264981921}{626233982400}} = 1.1672762409 \dots < 1.1673.$$

Sections 2–6 give this route. Its only scalar polynomial inequality is $(x - 1)^4 \geq 0$; it already illustrates the common-cone geometry, Gaussian transport, entropy, Hermite expansion, tensor symmetry, and the new matrix budget. The spectral refinement in Section 4 can be skipped on a first reading of the small-parameter proof. Section 6.4 then gives the much sharper bound 1.166474 using just one explicit quartic square and the original entropy estimate. Thus the spectral matrix constraint substantially sharpens the small-parameter bound without sixth moments, septic tests, or moment flow.

Here is the geometric outline. Lift a thin radial shell to an acute spherical set and complete its cone to one self-dual cone D . Partition the vertices by their latitude relative to the mean of $R = 2P_D - I$. A smoothed reflection changes a Gaussian vector before a vertex-dependent translation makes it a strict local maximum at the chosen vertex. Relative entropy bounds the Gaussian measure of the resulting normal cone, and sampling gives the partition. All vertices in a part use the same cone and the same Hermite data.

The first new point is that the higher Hermite corrections cannot be placed arbitrarily in matrix space. Write $J = DR$, let $L = J_1$, and put $S = \mathbb{E}L^2$. Instead of using only $S \preceq I$, we retain $S + \mathbb{E}J_3^2 \preceq I$. The fully symmetric coefficient tensors allow this budget to be applied at different indices of the cubic test. This rules out concentration of the test in directions already occupied by the first chaos. A weighted Cauchy–Schwarz refinement also retains the spectrum of S .

The second new point is an entropy estimate across smoothing scales. For $F_q = P_{\sqrt{q}}J$, the moments $M_r(q) = N^{-1} \mathbb{E} \operatorname{tr} F_q^{2r}$ obey an explicit differential inequality. Its fourth-moment version has a negative commutator correction. Antisymmetrizing a first-chaos projection of $[F_q, \partial_i F_q]$ supplies a lower bound for this correction from the original matrix series L . These identities preserve information lost by a single-scale Poincaré estimate.

The final certificate combines these ingredients with the sixth-moment Hessian projection and a joint septic test. All displayed parameters are rational, the scalar polynomial is positive on the entire nonnegative half-line, and the infinite Hermite tail is bounded analytically. The accompanying program checks the numerical part with integer fractions; it does not replace the analytic proofs.

Finally, Section 13 gives two geometric refinements. Keeping the original acute cone yields an explicit gain when its statistical dimension is below $N/2$. Choosing among correlated Gaussian proposals is also legitimate, thanks to concentration of the logarithmic Jacobian, but the pairwise-geometry relaxation admits the current worst profile. Section 14 separates such obstructions from lower bounds for the actual Borsuk problem.

2. CONE REFLECTIONS AND THEIR HERMITE TENSORS

Let $D \subset \mathbb{R}^N$ be a closed convex self-dual cone. We use the positive dual convention $D^* = \{y : \langle y, x \rangle \geq 0 \text{ for all } x \in D\}$. Moreau's decomposition [6] gives

$$x = P_D x - P_D(-x), \quad P_D x \perp P_D(-x).$$

Consequently $R = 2P_D - I$ is even, positively homogeneous, norm preserving, and 1-Lipschitz. Moreover

$$R(x) \pm x \in D, \quad R(x) \in D, \quad R = \nabla \phi, \quad \phi(x) = |P_D x|^2 - |x|^2/2.$$

The almost-everywhere Jacobian $J = DR$ is an odd symmetric contraction. These assertions follow from firm nonexpansiveness of projection and the differentiability of its squared norm.

Let G, Z be independent standard Gaussian vectors in $\mathbb{R}^N$. We use the Gaussian smoothing convention

$$(P_\rho f)(x) = \mathbb{E} f(\rho x + \sqrt{1 - \rho^2} Z), \quad 0 \leq \rho \leq 1.$$

Since R is even, write its orthogonal Hermite expansion as

$$R = \mu + \sum_{j \geq 1} R_{2j}, \quad \mu = \mathbb{E}R(G), \quad a_{2j} = \frac{1}{N} \mathbb{E}|R_{2j}(G)|^2.$$

Put $h = |\mu|^2/N$. Gaussian orthogonality and the derivative identity for Hermite chaoses [9] imply

$$h + \sum_{j \geq 1} a_{2j} = 1, \quad \sum_{j \geq 1} 2ja_{2j} \leq 1, \quad \frac{1}{2} \leq h \leq 1. \quad (1)$$

The first matrix chaos of J is

$$L(G) = \sum_{i=1}^N G_i A_i, \quad T_{ijk} = (A_i)_{jk}, \quad S = \sum_i A_i^2.$$

The tensor T is fully symmetric. More generally, the coefficient tensor of the k th matrix chaos of a Hessian is symmetric in all $k+2$ indices. These facts can be read from the Hermite expansion of the potential ϕ , or proved by Gaussian integration by parts. For a fixed vector v , orthogonal projection onto the first chaos gives $\mathbb{E}|L(G)v|^2 \leq \mathbb{E}|J(G)v|^2 \leq |v|^2$. Thus

$$S \preceq I, \quad m := N^{-1} \operatorname{tr} S = 2a_2. \quad (2)$$

Throughout, tensor norms are Euclidean norms with all ordered indices summed. The degree- k Hermite isometry is $\|I_k(U)\|_{L^2}^2 = k! \|U\|^2$ for a symmetric coefficient tensor U .

Let X be the absolute value of an eigenvalue of $L(G)$, chosen uniformly among its N eigenvalues, independently after G is sampled. Then

$$\mathbb{E}X^2 = m, \quad \mathbb{E}X \geq m. \quad (3)$$

Indeed, $N^{-1} \mathbb{E} \operatorname{tr}(LJ) = m$, while $\operatorname{tr}(LJ) \leq \operatorname{tr}|L|$ because J is a symmetric contraction.

3. THE ELEMENTARY HESSIAN ESTIMATE

We give the tensor estimate before eliminating the scalar moments. It supplies the framework for the matrix-budget improvement in the next section. Write

$$t = N^{-1} \mathbb{E} \operatorname{tr} L^4, \quad v = N^{-1} \operatorname{tr} S^2, \quad \delta = t - \frac{3}{2}v.$$

Define the fully symmetric quartic tensor

$$B_{abcd} = \sum_i T_{iab} T_{icd}, \quad H_{abcd} = B_{abcd} + B_{acbd} + B_{adbc}. \quad (4)$$

Wick's formula gives, with $Q_4 = \sum_{i,j} \operatorname{tr}(A_i A_j A_i A_j)$,

$$Nt = 2 \operatorname{tr} S^2 + Q_4, \quad \|H\|^2 = 3 \operatorname{tr} S^2 + 6Q_4 = 6N\delta. \quad (5)$$

In particular $\delta \geq 0$ and $t \geq \frac{3}{2}v \geq \frac{3}{2}m^2$.

Let Π_3 denote orthogonal projection of third-chaos matrix fields onto the subspace whose five-index coefficient tensors are fully symmetric, and put

$$Z = N^{-1} \|\Pi_3(L^3)_3\|_{L^2}^2.$$

Here $(L^3)_3$ is the third-chaos component of the matrix product L^3 .

Lemma 3.1. *One has $Z \leq 4\delta \leq 4t - 6m^2$.*

Proof. Before full symmetrization, the coefficient tensor is the path

$$U_{ab,c,de} = \sum_{p,q} T_{abp} T_{pcq} T_{qde}.$$

Attach one copy of T to H :

$$V_{abcde} = \sum_p H_{abcp} T_{pde} = U_{ab,c,de} + U_{ac,b,de} + U_{bc,a,de}.$$

Hence $\text{Sym}_5 V = 3 \text{Sym}_5 U$. The flattening $F : \mathbb{R}^N \rightarrow \text{Sym}^2 \mathbb{R}^N$, $(Fu)_{de} = \sum_p T_{pde} u_p$, satisfies $F^* F = S$ by full symmetry. It is a contraction by (2). Thus $\|V\| \leq \|H\|$, and

$$Z = \frac{3!}{N} \|\text{Sym}_5 U\|^2 \leq \frac{2}{3N} \|H\|^2 = 4\delta.$$

Finally $v \geq m^2$. $\square$

The estimate becomes a constraint on Hermite degrees by pairing with J . Since L^3 has only chaoses one and three, and $J_1 = L$,

$$\frac{1}{N} \mathbb{E} \text{tr}(JL^3) = t + \frac{1}{N} \langle J_3, \Pi_3(L^3)_3 \rangle_{L^2}.$$

Here $J_3 = DR_4$, so $N^{-1} \|J_3\|_{L^2}^2 = 4a_4$. Contraction gives $\text{tr}(JL^3) \leq \text{tr}|L|^3$. Consequently

$$(t - \mathbb{E}X^3)_+^2 \leq 4a_4 Z \leq 16a_4(t - 3m^2/2). \quad (6)$$

This provides the cubic test used by the matrix-budget refinement below.

4. THE MATRIX DERIVATIVE BUDGET

The first and third Hermite components of the Jacobian spend their contraction budget in the same directions. Keeping this operator restriction gives a nonlinear improvement of the cubic test. Write

$$J_3 = I_3(D), \quad B_3 = \mathbb{E}J_3^2, \quad A = \frac{1}{N} \text{tr} B_3 = 4a_4.$$

The coefficient tensor D is fully symmetric in its five indices. Orthogonality of the Hermite chaoses, applied to every fixed input vector, strengthens (2) to

$$S + B_3 \preceq I. \quad (7)$$

Retain H from (4), and introduce its marginal

$$\mathcal{Q}_{pp'} = \sum_{a,b,c} H_{abcp} H_{abcp'}, \quad R_0 = \frac{\text{tr} \mathcal{Q}}{9N} = \frac{2}{3}t - v, \quad y = \frac{\text{tr}(S\mathcal{Q})}{9N}.$$

Then $0 \leq y \leq R_0$ and the grafting argument gives $Z \leq 6y$. The marginal also satisfies

$$0 \preceq \mathcal{Q} \preceq 9\Phi(S) \preceq 9S, \quad \Phi(B) = \sum_i A_i B A_i.$$

Indeed, the three terms in $H(x, \cdot, \cdot, \cdot)$ each have squared norm $x^\top \Phi(S)x$; Cauchy–Schwarz gives the factor nine. This observation will also be used in the spectral refinement below.

Define

$$f(A) = \begin{cases} A - \frac{3}{4}A^2, & 0 \leq A \leq 2/3, \\ 1/3, & A \geq 2/3. \end{cases}$$

This is an increasing concave function on $[0, \infty)$.

Theorem 4.1. *Let $g = (t - \mathbb{E}X^3)_+$. Then*

$$g^2 \leq 8y \left(1 - \frac{y}{R_0}\right), \quad (8)$$

where the right side is zero when $R_0 = 0$. Consequently

$$g^2 \leq 6R_0 f(A) \leq 4 \left(t - \frac{3}{2}m^2\right) f(A). \quad (9)$$

In particular, for every $A_0 \in [0, 2/3)$ and $\gamma \geq 0$,

$$A \geq \frac{2\gamma(t - \mathbb{E}X^3) - 4\gamma^2(t - \frac{3}{2}m^2) - \frac{3}{4}A_0^2}{1 - \frac{3}{2}A_0}. \quad (10)$$

Proof. As in the proof of Lemma 3.1, form

$$V_{abcde} = \sum_p H_{abcp} T_{pde}, \quad \text{Sym}_5 V = 3 \text{Sym}_5 U.$$

Pairing the cubic test with $J_3 = I_3(D)$ and using full symmetry of D gives $g \leq 2|\langle D, V \rangle|/N$. Flatten both tensors along the index a . The Hermite isometry and full symmetry of D give $DD^* = B_3/6$. Thus (7) implies $D = (I - S)^{1/2}K/\sqrt{6}$ for a rectangular contraction K . By nuclear-norm duality and $\|M\|_* \leq \sqrt{N}\|M\|_{\text{HS}}$,

$$|\langle D, V \rangle|^2 \leq \frac{N}{6} \langle V, (I - S)_a V \rangle = \frac{N}{6} \langle H, (I - S)_a S_p H \rangle.$$

The last identity contracts the two copies of T in the indices d, e . Hence

$$g^2 \leq \frac{2}{3N} \langle H, (I - S)_1 S_4 H \rangle. \quad (11)$$

For operators acting on the indicated tensor indices, put

$$q = \frac{\|H\|^2}{N} = 9R_0, \quad r = \frac{\langle H, S_1 H \rangle}{N} = 9y, \quad b = \frac{\langle H, S_1^2 H \rangle}{N}, \quad c = \frac{\langle H, S_1 S_2 H \rangle}{N}.$$

If $q = 0$, the assertion is immediate. Otherwise, full symmetry of H and Cauchy–Schwarz imply

$$4b + 12c = \frac{1}{N} \left\| \sum_{i=1}^4 S_i H \right\|^2 \geq \frac{16r^2}{q}.$$

Since $0 \preceq S \preceq I$, one has $b \leq r$ and therefore $c \geq (4r^2/q - r)/3$. Inserting this in (11) proves

$$g^2 \leq \frac{2}{3}(r - c) \leq \frac{8}{9} \left(r - \frac{r^2}{q} \right) = 8y \left(1 - \frac{y}{R_0} \right).$$

The ordinary cubic test also gives $g^2 \leq AZ \leq 6Ay$. For $A \leq 2/3$, the maximum over $y \in [0, R_0]$ of $\min\{6Ay, 8y(1 - y/R_0)\}$ occurs at $y = R_0(1 - 3A/4)$ and equals $6R_0(A - 3A^2/4)$. For $A \geq 2/3$, the maximum is $2R_0$. This proves the first inequality in (9); the second uses $v \geq m^2$.

Concavity gives $f(A) \leq (1 - 3A_0/2)A + 3A_0^2/4$ for every $A \geq 0$. For $d = t - 3m^2/2 > 0$, the identity $g^2/(4d) = \sup_{\gamma \geq 0} \{2\gamma g - 4\gamma^2 d\}$ then proves (10), using $t - \mathbb{E}X^3 \leq g$. If $d = 0$, then $g = 0$ and the same inequality follows directly. $\square$

The improvement uses two instances of symmetry: the fifth-order coefficient tensor spends the same matrix budget in each index, and the four indices of H have identical marginals. The previous scalar bound allowed its quartic marginal to concentrate in directions where S already exhausted the contraction budget. Inequality (8) rules out that configuration whenever the cubic-test gap is positive.

4.1. A spectral refinement of the matrix budget. The same argument can retain one more marginal moment. Put

$$Y = \mathcal{Q}/9, \quad z = \frac{\text{tr}(S^2 Y)}{N}.$$

As before, $R_0 = \text{tr} Y/N$, $y = \text{tr}(SY)/N$ and $g = (t - \mathbb{E}X^3)_+$. Before using $S^2 \preceq S$ in the proof of Theorem 4.1, one obtains

$$g^2 \leq 6y + 2z - \frac{8y^2}{R_0}. \quad (12)$$

The marginal inequality $0 \preceq Y \preceq S \preceq I$ gives, for every real λ ,

$$0 \leq \frac{1}{N} \operatorname{tr}((I - Y)(S - \lambda I)^2) = v - z - 2\lambda(m - y) + \lambda^2(1 - R_0).$$

Thus

$$z \leq v - 2\lambda(m - y) + \lambda^2(1 - R_0). \quad (13)$$

Lemma 4.2. *Fix real parameters λ, c such that*

$$d = 16c - 6 - 4\lambda > 0, \quad 2 + 2\lambda^2 - 8c^2 \leq 0,$$

and define

$$F_{\lambda,c}(m, t) = \frac{16c^2 - 4\lambda^2}{3} t + (2 + 2\lambda^2 - 8c^2)m^2 - 4\lambda m + 2\lambda^2. \quad (14)$$

Then $F_{\lambda,c}(m, t) \geq 0$ and

$$g^2 \leq \frac{6A}{6A + d} F_{\lambda,c}(m, t). \quad (15)$$

For every $A_0 \geq 0$ and $\gamma \geq 0$, this implies the global affine constraint

$$A \geq \frac{2\gamma(t - \mathbb{E}X^3) - \gamma^2 F_{\lambda,c}(m, t) - \beta}{\alpha}, \quad \alpha = \frac{6d}{(6A_0 + d)^2}, \quad \beta = \frac{36A_0^2}{(6A_0 + d)^2}. \quad (16)$$

Proof. Insert (13) into (12), and apply $-y^2/R_0 \leq -2cy + c^2R_0$. This gives

$$\begin{aligned} g^2 &\leq (6 + 4\lambda - 16c)y + (2 + 2\lambda^2 - 8c^2)v \\ &\quad + \frac{16c^2 - 4\lambda^2}{3} t - 4\lambda m + 2\lambda^2. \end{aligned}$$

Because the coefficient of v is nonpositive and $v \geq m^2$, the right side is at most $F_{\lambda,c}(m, t) - dy$. Combining this with $g^2 \leq 6Ay$ proves (15) for $A > 0$; the case $A = 0$ follows from $g = 0$. The inequality $F_{\lambda,c}(m, t) \geq 0$ also follows from $g^2 + dy \leq F_{\lambda,c}(m, t)$. Finally, the function $6A/(6A + d)$ is concave, and its tangent at A_0 is $\alpha A + \beta$. The scalar variational inequality $2\gamma g - \gamma^2 F \leq g^2/F$ proves (16). The case $F = 0$ is immediate. $\square$

5. THE COMMON-CONE TRANSPORT

We give the geometric and entropy argument in full, including the reductions needed for a uniform Borsuk bound. As in the elementary note [11], a direction selects vertices at which it defines a strict local maximum. Here one cone is chosen for an entire spherical part, so all its vertices use the same reflection and Hermite data. For a graph represented by distinct vectors, let

$$\mathcal{K}_x = \{u : \langle u, x - y \rangle > 0 \text{ for every neighbor } y\}, \quad I_u = \{x : u \in \mathcal{K}_x\}.$$

Each I_u is independent. If, for each vertex x , a random u lies in $\mathcal{K}_x$ with probability at least q , then $\lceil (\log |V| + 1)/q \rceil$ sampled directions cover all vertices with positive probability. Assigning a vertex to its first such set gives a coloring.

5.1. Reduction to a common mean level. After taking the closure, center a minimum enclosing ball of a diameter-one set at the origin. Its radius satisfies $R \leq 1/\sqrt{2}$: contact points v_i admit weights λ_i with $\sum_i \lambda_i v_i = 0$ and

$$2R^2 = \sum_{i,j} \lambda_i \lambda_j |v_i - v_j|^2 \leq 1.$$

A finite ε -net has cardinality at most $(1 + 2/\varepsilon)^n$. Divide it into radial intervals of width ε , and in each interval join pairs whose distance is at least $1 - 4\varepsilon$. An independent set extends to a set of diameter at most $1 - 2\varepsilon$ in the original set.

On an exact sphere $|x| = r \leq 1/\sqrt{2}$, the lift

$$\hat{x} = (\sqrt{2}x, \sqrt{1-2r^2}) \in \mathbb{S}^n$$

satisfies $\hat{x} \cdot \hat{y} = 1 - |x - y|^2 \geq 0$. Thus diameter neighbors become orthogonal. The cone of all lifted vertices is acute and is contained in a self-dual cone D , by the completion theorem of Barker and Foran [5]. For a thin shell $r_- \leq |x| \leq r_+$, use (x, g) with $g^2 = 1/2 - r_-^2$ and normalize individually. These lifts remain acute; adjacent vertices have inner product $O(\varepsilon + r_+^2 - r_-^2)$. This change of representation preserves independence of strict local maxima.

Put $N = n + 1$. Choose D before partitioning the vertices into intervals of width η according to

$$\alpha = \langle f, x \rangle, \quad f = \mu/|\mu|.$$

Since $x, \mu \in D$, one has $0 \leq \alpha \leq 1$. There are only $O_\eta(1)$ parts. We first work on an exact level with orthogonal edges; the uniform perturbation is addressed below.

5.2. Two common constraints on a neighbor. For $0 < \rho, \sigma \leq 1$, write

$$R_\rho(x) = \mathbb{E}R(\rho x + \sqrt{1-\rho^2}G'), \quad F_\sigma = \sigma^{-2}(R_\sigma - \mu),$$

where G' is independent Gaussian. Choose $0 \leq s < \sigma$ and $\beta \geq 0$, and set

$$\mathcal{T}(x) = x - sF_\sigma(x), \quad K_\rho(x) = R_\rho(x) - \rho x \in D, \quad \mathcal{H} = \mathcal{T} + \beta K_\rho - \beta \mu.$$

Define

$$\begin{aligned} V &= \sum_{j \geq 1} \sigma^{4j-4} a_{2j}, & U &= \sum_{j \geq 1} (s\sigma^{2j-2} - \beta\rho^{2j})^2 a_{2j}, \\ Q_0 &= (1 - \beta\rho)^2 + U, & W &= \sum_{j \geq 1} 2j\sigma^{4j-4} a_{2j}. \end{aligned} \tag{17}$$

Evenness gives $\mathbb{E}\mathcal{H} = 0$, $\mathbb{E}|\mathcal{H}|^2 = NQ_0$, and $\mathbb{E}|\mathcal{T}|^2 = N(1 + s^2V)$. All relevant Lipschitz constants are bounded independently of N .

For a vertex x , write $w = 1 - \alpha^2$ and $x = \alpha f + \sqrt{w}e$, where $e \perp f$. An orthogonal neighbor at the same mean level has the form

$$y = \alpha f - \frac{\alpha^2}{\sqrt{w}}e + y_0, \quad y_0 \perp f, e, \quad |y_0|^2 = \frac{2w-1}{w}. \tag{18}$$

Hence a nonisolated level has $w \geq 1/2$. Gaussian Poincare, applied to fixed scalar components and to the norm of $\mathcal{H}$, gives with probability $1 - o(1)$, uniformly in the vertex,

$$\mathcal{H} \cdot f = o(\sqrt{N}), \quad \mathcal{H} \cdot e = o(\sqrt{N}), \quad |\mathcal{H}| \leq (\sqrt{Q_0} + o(1))\sqrt{N}.$$

Because $K_\rho \cdot y \geq 0$, this controls all neighbors simultaneously:

$$\mathcal{T}(G) \cdot y \leq \left(\sqrt{\frac{2w-1}{w}}Q_0 + \beta\sqrt{h(1-w)} + o(1) \right) \sqrt{N}.$$

Let $z = x - \alpha f$. Then $|z|^2 = w$ and $z \cdot (x - y) = 1$. The fixed component $\mathcal{T}(G) \cdot x$ has bounded variance. Adding to $\mathcal{T}(G)$ the vector $(S_0 + o(1))\sqrt{N}z$, with

$$S_0 = \sqrt{\frac{2w-1}{w}}Q_0 + \beta\sqrt{h(1-w)},$$

therefore produces a random vector $Y \in \mathcal{K}_x$ with probability $1 - o(1)$. The positive $o(1)$ may be chosen uniformly, for example of order $N^{-1/8}$ in the exact geometry. Its normalized second moment is

$$N^{-1}\mathbb{E}|Y|^2 = 1 + s^2V + \left(\sqrt{(2w-1)Q_0} + \beta\sqrt{hw(1-w)} \right)^2 + o(1). \tag{19}$$

For $0 < \zeta \leq 1$, put

$$A = \frac{1 + \zeta^2}{2\zeta}, \quad B = \frac{1 + \zeta^2}{4}. \quad (20)$$

For $t = 2w - 1 \in [0, 1]$, weighted Cauchy–Schwarz and

$$\frac{t}{A} + \frac{1 - t^2}{4B} = 1 - \frac{(t - \zeta)^2}{1 + \zeta^2} \leq 1$$

bound (19) by $\mathcal{M} + o(1)$, where

$$\mathcal{M} = 1 + A(1 - \beta\rho)^2 + s^2V + AU + B\beta^2h. \quad (21)$$

5.3. Entropy cost and uniformity. The derivative $J_\sigma = DF_\sigma$ is odd, symmetric, and satisfies $\|J_\sigma\|_{\text{op}} \leq 1/\sigma$. The map $\mathcal{T}$ is globally bi-Lipschitz, since its inverse equation is a contraction. Pairing G and $-G$ gives

$$\mathbb{E} \log \det(I - sJ_\sigma) = \frac{1}{2} \mathbb{E} \text{tr} \log(I - s^2J_\sigma^2) \geq -\frac{N}{2}cW, \quad c = -\sigma^2 \log(1 - s^2/\sigma^2). \quad (22)$$

The inequality uses $\log(1 - ru) \geq u \log(1 - r)$ for $0 \leq u \leq 1$. Let ν be the law of Y rescaled to have second moment N . Change of variables [7] yields

$$\text{Ent}(\nu \mid \gamma_N) = \frac{N}{2} \log \frac{\mathbb{E}|Y|^2}{N} - \mathbb{E} \log \det(I - sJ_\sigma) \leq \frac{N}{2}(\log \mathcal{M} + cW) + o(N).$$

Scaling preserves $\mathcal{K}_x$. Since $\nu(\mathcal{K}_x) = 1 - o(1)$, the binary relative-entropy inequality implies

$$\gamma_N(\mathcal{K}_x) \geq \exp \left[-\frac{N}{2}(\log \mathcal{M} + cW) + o(N) \right]. \quad (23)$$

For completeness, the preceding estimates are stable under the two coarse partitions. If $0 \leq x \cdot y \leq \epsilon$ and $|f \cdot (x - y)| \leq \eta$, the residual squared norm in (18) changes by $O(\epsilon + \eta)$, and $z \cdot (x - y) = 1 + O(\epsilon + \eta)$. A nonisolated part has $w \geq 1/2 - O(\epsilon + \eta)$, so denominators stay bounded. The support estimate changes by at most $O(\sqrt{\epsilon + \eta})\sqrt{N}$, uniformly in D and x . The concentration estimates have uniform dimension-independent Lipschitz bounds. Thus, for fixed partition widths, the normalized entropy changes by $o_{\epsilon, \eta}(1) + o_N(1)$. First take $n \rightarrow \infty$ and then let the widths tend to zero.

It follows from the sampling argument that a uniform estimate $\log \mathcal{M} + cW \leq 2 \log C$ implies $\limsup b(n)^{1/n} \leq C$. The palettes for shells and mean levels cost only a constant for fixed widths; $\log |V| = O_\epsilon(n)$ costs only a polynomial factor.

5.4. A scalar form of the transport criterion. Set

$$\begin{aligned} M_0 &= 1 + A(1 - \beta\rho)^2 + B\beta^2, \\ b_j &= s^2\sigma^{4j-4} + A(s\sigma^{2j-2} - \beta\rho^{2j})^2 - B\beta^2. \end{aligned} \quad (24)$$

Then $\mathcal{M} = M_0 + \sum_j b_j a_{2j}$. For any $M > 0$, the tangent inequality for the logarithm gives

$$\log \mathcal{M} + cW \leq \log M - 1 + \frac{M_0}{M} + \sum_{j \geq 1} d_j a_{2j}, \quad d_j = \frac{b_j}{M} + 2jc\sigma^{4j-4}. \quad (25)$$

6. A SHORT RATIONAL CERTIFICATE

The numerical optimization can be separated from the geometric and analytic ideas. Here is a version using small rational parameters and the elementary scalar inequality $(x-1)^4 \geq 0$, together with the new Hessian matrix budget:

$$\limsup_{n \rightarrow \infty} b(n)^{1/n} \leq \sqrt{\frac{853264981921}{626233982400}} = 1.1672762409 \dots < 1.1673. \quad (26)$$

This bound uses the common cone and common mean level, the centered Gaussian transport, the entropy change of variables, and the fully symmetric Hessian estimate and its matrix-budget refinement. The sharper entropy estimate below is compatible with the same certificate, though its improvement is not needed to simplify the final fraction.

6.1. A particularly simple fourth-Hermite constraint. Retain $m = 2a_2$, $t = \mathbb{E}X^4$, $\Delta = t - 3m^2/2 \geq 0$, and put $A_4 = 4a_4$. Define the nondecreasing concave cap

$$\Phi(A) = \begin{cases} A - 3A^2/4, & 0 \leq A \leq 2/3, \\ 1/3, & A \geq 2/3. \end{cases}$$

The matrix-budget estimate (9), including its uneliminated line-parabola form, gives

$$(t - \mathbb{E}X^3)_+^2 \leq 4\Delta\Phi(A_4).$$

For $A_4 \geq 2/3$, this follows by maximizing the parabola: $(t - \mathbb{E}X^3)_+^2 \leq 2R_0 \leq 4\Delta/3$, where $R_0 = 2t/3 - v$ and $v \geq m^2$ are as in the matrix-budget estimate. The scalar inequality $(X-1)^4 \geq 0$, together with $\mathbb{E}X^2 = m$ and $\mathbb{E}X \geq m$, gives

$$\mathbb{E}X^3 \leq \frac{t}{4} + \frac{m}{2} + \frac{1}{4}.$$

Consequently, with $K = 9m^2/8 - m/2 - 1/4$,

$$(3\Delta/4 + K)_+^2 \leq 4\Delta\Phi(A_4).$$

If $K \leq 0$, then $\Phi(A_4) \geq 3K/4$ is automatic. If $K > 0$, then $\Delta > 0$ and $(3\Delta/4 + K)^2 \geq 3K\Delta$ gives the same conclusion. Thus

$$\Phi(A_4) \geq B(m) := \frac{27}{32}m^2 - \frac{3}{8}m - \frac{3}{16}.$$

Two rational tangents finish the calculation. Convexity of B , with contact at $m = 5/6$, and concavity of Φ , with contact at $A = 1/8$, give respectively

$$B(m) \geq \frac{33}{32}m - \frac{99}{128}, \quad \Phi(A) \leq \frac{13}{16}A + \frac{3}{256}.$$

The second inequality holds on the entire capped domain, not just $A \leq 2/3$. Combining the two yields

$$\boxed{4a_4 \geq \frac{33}{26}(2a_2) - \frac{201}{208}.} \quad (27)$$

6.2. A choice of transport parameters. Use the transport and entropy criterion already proved, with

$$\beta = \frac{4}{5}, \quad \rho = \frac{5}{6}, \quad s = \frac{1}{8}, \quad \sigma = \frac{3}{4}, \quad \zeta = \frac{2}{3}, \quad M = \frac{4}{3}, \quad \omega = \frac{1}{120}.$$

Thus

$$A = \frac{13}{12}, \quad B = \frac{13}{36}, \quad M_0 = 1 + A(1 - \beta\rho)^2 + B\beta^2 = \frac{3649}{2700}.$$

Writing $u = s^2/\sigma^2 = 1/36$, the elementary inequality $-\log(1-u) \leq u/(1-u)$ gives

$$c = -\sigma^2 \log(1 - s^2/\sigma^2) \leq \frac{9}{560} =: \bar{c}.$$

For

$$b_j = s^2 \sigma^{4j-4} + A(s\sigma^{2j-2} - \beta\rho^{2j})^2 - B\beta^2, \quad \bar{d}_j = b_j/M + 2j\bar{c}\sigma^{4j-4},$$

direct rational arithmetic gives

$$\bar{d}_1 = \frac{306967}{14515200} < \frac{11}{520} = 2\omega\frac{33}{26}, \quad \bar{d}_2 = -\frac{20591657753}{300987187200} < -\frac{1}{30} = -4\omega.$$

There is a short uniform check for every remaining degree. After omitting the negative cross term in the square defining b_j ,

$$M\bar{d}_j \leq [(1+A)s^2 + 2jM\bar{c}]\sigma^{4j-4} + A\beta^2\rho^{4j} - B\beta^2.$$

For $j \geq 3$,

$$\sigma^{4j-4} < \frac{1}{9}, \quad j\sigma^{4j-4} < \frac{1}{3}, \quad \rho^{4j} < \frac{1}{8}.$$

The middle sequence decreases from $j = 3$, since its successive ratio is at most $(4/3)(81/256) < 1$. The other two bounds are immediate. The positive terms in the preceding envelope are therefore less than

$$\frac{25}{6912} + \frac{1}{70} + \frac{13}{150} < \frac{1}{8} < \frac{52}{225} = B\beta^2.$$

It follows that $\bar{d}_j < 0$ for every $j \geq 3$.

The tangent inequality for the logarithm and (27) give

$$\begin{aligned} \log \mathcal{M} + cW &\leq \log M - 1 + M_0/M + \omega \left(\frac{33}{26}(2a_2) - 4a_4 \right) \\ &\leq \log(4/3) + x, \quad x = \frac{8111}{374400}. \end{aligned}$$

For $0 \leq x < 3$, the elementary bound $k! \geq 2 \cdot 3^{k-2}$ for $k \geq 2$ gives

$$e^x \leq 1 + x + \frac{x^2}{2(1-x/3)}.$$

The squared exponential base is therefore at most

$$\frac{4}{3} \left(1 + x + \frac{x^2}{2(1-x/3)} \right) = \frac{853264981921}{626233982400} < \left(\frac{11673}{10000} \right)^2.$$

This proves (26). All displayed comparisons are rational, and the tail argument covers every Hermite degree.

6.3. Where the improved entropy inequality enters. For clarity, the new fourth-Jacobian-moment estimate can also be inserted at the entropy step above. Put $F = P_\sigma J$, $q = \sigma^2$, and

$$\kappa = \frac{q(3-q)}{1+q}.$$

The estimate $\mathbb{E} \operatorname{tr} F^4 \leq \kappa \mathbb{E} \operatorname{tr} F^2$ improves the determinant coefficient to

$$c_{\text{new}} = (1-\kappa)s^2 + \kappa c \leq c.$$

For the small rational parameters used here,

$$q = \frac{9}{16}, \quad \kappa = \frac{351}{400}, \quad c_{\text{new}} \leq \frac{14351}{896000} < \frac{9}{560}.$$

Thus all the displayed dual inequalities continue to hold after this strict improvement. Keeping the larger value $9/560$ makes the certificate more transparent; the moment flow and commutator correction below supply further improvements used in the sharper theorem.

6.4. A sharper quartic certificate. The new spectral matrix-budget inequality is useful without the sixth-moment projection, the septic test, or the refined entropy hierarchy. The following rational specialization uses only one explicit quartic square and the original logarithmic Jacobian estimate. Take

$$\gamma = \frac{9}{20}, \quad \lambda_b = \frac{5}{6}, \quad c_b = \frac{8}{9}, \quad A_0 = \frac{1}{9}, \quad m_0 = \frac{5}{6}.$$

Then

$$d_b = \frac{44}{9}, \quad \alpha_b = \frac{594}{625}, \quad \beta_b = \frac{9}{625}, \quad F_4 = \frac{799}{243}, \quad F_2 = -\frac{475}{162}.$$

Put $k_4 = 2\gamma - \gamma^2 F_4 = 281/1200$ and $a = 540/281$, $b = -4/5$. Define

$$c_0 = -k_4 b^2, \quad c_1 = -2abk_4, \quad c_2 = -k_4(a^2 - 2b).$$

The exact identity

$$k_4 x^4 - 2\gamma x^3 - c_0 - c_1 x - c_2 x^2 = \frac{281}{1200} \left(x^2 - \frac{540}{281} x + \frac{4}{5} \right)^2 \geq 0$$

and $c_1 > 0$, $\mathbb{E}X \geq m$ give

$$\frac{594}{625} A_4 \geq f_0 + f_1 m + f_2 m^2,$$

where

$$f_0 = c_0 - 2\gamma^2 \lambda_b^2 - \beta_b, \quad f_1 = c_1 + c_2 + 4\gamma^2 \lambda_b, \quad f_2 = -\gamma^2 F_2 > 0.$$

Taking the quadratic tangent at $m_0 = 5/6$ gives the particularly simple constraint

$$\frac{594}{625} 4a_4 \geq \frac{3861439}{3372000} m - \frac{617647}{720000}. \quad (28)$$

No scalar moment above order four is needed for this step.

For the transport use the exact decimals

β	0.7827996394	ρ	0.8491703718
s	0.1249486022	σ	0.7381553257
ζ	0.6785218289	M	1.3420283373
ω	0.0137606737		

Put $q = \sigma^2$, $u = s^2/q$ and use the rational upper bound

$$c_+ = q \left(\sum_{k=1}^8 \frac{u^k}{k} + \frac{u^9}{9(1-u)} \right) \geq -q \log(1-u).$$

Thus the unrefined doubled entropy cost is at most $\sum_{j \geq 1} 2j c_+ q^{2j-2} a_{2j}$. With $d_j = b_j/M + 2j c_+ q^{2j-2}$, the required dual inequalities are

$$d_1 \leq 2\omega \frac{3861439}{3372000}, \quad d_2 \leq -4\omega \frac{594}{625}, \quad d_j \leq 0 \quad (j \geq 3).$$

Exact rational arithmetic verifies the first two with margins exceeding 2.00×10^{-8} and 2.01×10^{-8} ; the remaining degrees $3 \leq j \leq 50$ have margin greater than 0.1045. The same decreasing geometric envelope as in the main certificate covers $j \geq 50$, with margin greater than 0.2237. Finally,

$$M \exp \left(\frac{M_0}{M} - 1 + \omega \frac{617647}{720000} \right) < (1.166474)^2.$$

This yields the independent bound

$$\limsup_{n \rightarrow \infty} b(n)^{1/n} < 1.166474.$$

The value from these parameters is approximately 1.16647305715. The accompanying program checks the parameter conditions, the finite and infinite spectral inequalities, and the exponential comparison using rational arithmetic only. The squared-base margin is greater than 2.19×10^{-6} .

7. HIGHER MOMENTS OF SMOOTHED JACOBIANS AND THE ENTROPY COST

We first record a useful fixed-scale Jacobian estimate. The moment flow in Section 8 will sharpen the entropy bound used in the final certificate. Gaussian smoothing makes an odd matrix contraction strictly smaller in successive even moments. At the fourth moment, one can also retain the defect $m - v = N^{-1} \operatorname{tr}(S - S^2)$. The proof has three steps: a pointwise reverse Poincare inequality controls the derivatives of the smoothed field, parity improves the Gaussian Poincare inequality for its even powers, and a matrix Cauchy–Schwarz inequality controls their means. The resulting moment bounds can then be summed in the logarithmic Jacobian expansion.

Proposition 7.1. *Let $J : \mathbb{R}^N \rightarrow \mathbb{R}^{N \times N}$ be a measurable odd symmetric matrix field with $\|J(x)\|_{\text{op}} \leq 1$ almost everywhere. Write its first Gaussian chaos as $L(G) = \sum_i G_i A_i$, and set*

$$S = \sum_i A_i^2, \quad m = \frac{1}{N} \operatorname{tr} S, \quad v = \frac{1}{N} \operatorname{tr} S^2.$$

For $0 < \sigma < 1$, let P_σ denote Gaussian smoothing and put

$$F = P_\sigma J, \quad q = \sigma^2, \quad k = \frac{q}{1-q}, \quad T_r = \frac{1}{N} \mathbb{E} \operatorname{tr} F(G)^{2r}.$$

Then $0 \preceq S \preceq I$ and

$$T_2 \leq \kappa T_1 - D(m - v), \quad \kappa = \frac{q(3-q)}{1+q}, \quad D = q^2(1-q)^2. \quad (29)$$

For every integer $r \geq 3$,

$$T_r \leq \kappa_r T_{r-1}, \quad \kappa_r = \begin{cases} \frac{r^2 k}{1 + r^2 k}, & r \text{ odd}, \\ \frac{q + r^2 k/2}{1 + r^2 k/2}, & r \text{ even}. \end{cases} \quad (30)$$

In particular $0 < \kappa_r < 1$.

Proof. Projection onto the first chaos gives $S \preceq \mathbb{E} J^2 \preceq I$. If $M_F = \mathbb{E} F^2$, odd-chaos orthogonality yields

$$qS \preceq M_F \preceq qS + q^3(I - S) \preceq qI. \quad (31)$$

Consequently $qI - M_F \succeq q(1 - q^2)(I - S)$, and traces against positive semidefinite matrices give

$$\begin{aligned} q \operatorname{tr} M_F - \operatorname{tr} M_F^2 &= \operatorname{tr}(M_F(qI - M_F)) \\ &\geq q \operatorname{tr}(S(qI - M_F)) \geq q^2(1 - q^2) \operatorname{tr}(S(I - S)). \end{aligned} \quad (32)$$

No commutation of M_F and S is needed.

Write $D_i = \partial_i F$. Differentiating the Gaussian smoothing formula and applying Bessel's inequality to each fixed vector gives

$$\sum_i D_i^2 \preceq k(P_\sigma(J^2) - F^2) \preceq k(I - F^2). \quad (33)$$

Indeed, for a fixed x and vector a , $D_i(x)a$ is $\sqrt{k}$ times the inner product of the centered vector-valued function $J(\sigma x + \sqrt{1-q}G')a$ with G'_i .

For any integer $r \geq 1$, differentiation of F^r gives

$$\sum_i \|\partial_i F^r\|_{\text{HS}}^2 \leq r^2 \operatorname{tr} \left(F^{2r-2} \sum_i D_i^2 \right) \leq r^2 k \operatorname{tr}(F^{2r-2} - F^{2r}). \quad (34)$$

For the first inequality, apply Cauchy–Schwarz to the r terms in each derivative. In an eigenbasis of F , weighted arithmetic–geometric mean and the symmetry of D_i bound each resulting squared norm by $\operatorname{tr}(F^{2r-2} D_i^2)$.

The Gaussian Poincare constant is $1/2$ for centered even functions. Apply this entrywise to $F^2 - M_F$, then use (34) and (32). This gives

$$NT_2 - \text{tr } M_F^2 \leq 2kN(T_1 - T_2), \quad (1 + 2k)T_2 \leq (q + 2k)T_1 - q^2(1 - q^2)(m - v),$$

which is (29).

For odd r , $\mathbb{E}F^r = 0$. Ordinary Poincare and (34) imply $T_r \leq r^2k(T_{r-1} - T_r)$. For even r , the block Gram matrix

$$\begin{pmatrix} \mathbb{E}F^2 & \mathbb{E}F^r \\ \mathbb{E}F^r & \mathbb{E}F^{2r-2} \end{pmatrix}$$

is positive semidefinite. By (31), replacing its upper-left block by qI preserves this property. Its Schur complement therefore gives

$$\|\mathbb{E}F^r\|_{\text{HS}}^2 \leq q \mathbb{E} \text{tr } F^{2r-2}.$$

Even Poincare and (34) now imply $T_r \leq qT_{r-1} + (r^2k/2)(T_{r-1} - T_r)$. Rearranging proves both cases of (30). $\square$

We apply the proposition to $J = DR$. For the transport in Section 5, one has $J_\sigma = F/\sigma$ and $T_1 = qW$. Put $u = s^2/q \in (0, 1)$ and

$$\mathcal{R}(u) = \sum_{r=2}^{\infty} \frac{u^r}{r} \prod_{j=3}^r \kappa_j, \quad (35)$$

where an empty product equals one. Pairing G and $-G$ and expanding the logarithm gives

$$-\frac{2}{N} \mathbb{E} \log \det(I - sJ_\sigma) = \sum_{r \geq 1} \frac{u^r T_r}{r} \leq uT_1 + \mathcal{R}(u)(\kappa T_1 - D(m - v)).$$

Thus the entropy bound improves to

$$\text{Ent}(\nu \mid \gamma_N) \leq \frac{N}{2} (\log \mathcal{M} + c_{\text{sharp}} W - e(m - v)) + o(N), \quad \begin{aligned} c_{\text{sharp}} &= s^2 + q\kappa\mathcal{R}(u), \\ e &= D\mathcal{R}(u). \end{aligned} \quad (36)$$

For an exact finite certificate, let

$$\mathcal{R}_8^+(u) = \sum_{r=2}^8 \frac{u^r}{r} \prod_{j=3}^r \kappa_j + \left(\prod_{j=3}^8 \kappa_j \right) \frac{u^9}{9(1-u)}. \quad (37)$$

Since $0 < \kappa_j < 1$, this bounds (35) from above and is rational whenever q and u are rational. It may replace $\mathcal{R}(u)$ simultaneously in c_{sharp} and e : the multiplying quantity $\kappa T_1 - D(m - v)$ is nonnegative by (29). Higher truncation orders work in the same way.

The moment hierarchy here uses only oddness, symmetry, and boundedness of J . The Hessian polynomial tests impose additional constraints on particular Hermite degrees. Both hierarchies are therefore available in the same optimization: one sharpens the logarithmic Jacobian cost, while the other restricts the admissible Hermite coefficients. For the certificate it is convenient to separate the two signs of the entropy correction. Let $\mathcal{R}_8^-$ be the sum through $r = 8$ in (37), without its final tail term. Set

$$c_+ = s^2 + q\kappa\mathcal{R}_8^+, \quad e_- = D\mathcal{R}_8^-. \quad (38)$$

Since $m - v \geq 0$ and $\mathcal{R}_8^- \leq \mathcal{R} \leq \mathcal{R}_8^+$, the normalized Jacobian cost is at most $c_+W - e_-(m - v)$.

8. MOMENT FLOW FOR THE SMOOTHED JACOBIAN

The moment estimates can be sharpened by following the entire Ornstein–Uhlenbeck smoothing trajectory, rather than applying a Poincare inequality only at its final time. The resulting bounds are linear in the separate Hermite energies and admit rational certificates.

Let J be an odd symmetric matrix field with $\|J(x)\|_{\text{op}} \leq 1$. For $0 < q < 1$, put

$$F_q = P_{\sqrt{q}}J, \quad M_r(q) = N^{-1} \mathbb{E} \operatorname{tr} F_q^{2r}.$$

The standard Gaussian semigroup identities used below may be found in [9, 8]; the trace arithmetic–geometric mean step is the same one used in matrix Gaussian moment estimates, cf. [13]. The new point is to retain the dependence on the smoothing parameter.

Lemma 8.1. *For every integer $r \geq 2$,*

$$M'_r(q) \leq \frac{\chi_r}{1-q} (M_{r-1}(q) - M_r(q)), \quad \chi_r = r(2r-1).$$

Consequently,

$$M_r(q) \leq \chi_r (1-q)^{\chi_r} \int_0^q \frac{M_{r-1}(t)}{(1-t)^{\chi_r+1}} dt.$$

Proof. Let $\mathcal{L} = \Delta - x \cdot \nabla$ be the Gaussian generator and $D_i = \partial_i F_q$. Since $\partial_q F_q = -\mathcal{L}F_q/(2q)$, integration by parts gives

$$M'_r(q) = \frac{r}{qN} \sum_i \sum_{j=0}^{2r-2} \mathbb{E} \operatorname{tr} (D_i F_q^j D_i F_q^{2r-2-j}).$$

In an eigenbasis of F_q , weighted arithmetic–geometric mean, followed by exchanging the two eigenvalue indices, bounds each summand by $\operatorname{tr}(D_i^2 F_q^{2r-2})$. This remains valid for negative eigenvalues: take absolute values before applying arithmetic–geometric mean. The conditional Gaussian Bessel inequality gives pointwise

$$\sum_i D_i^2 \preceq \frac{q}{1-q} (I - F_q^2).$$

Multiplication by the positive semidefinite matrix F_q^{2r-2} and tracing therefore proves the differential inequality. Oddness gives $F_0 = \mathbb{E}J = 0$, and bounded convergence gives $M_r(q) \rightarrow 0$ as $q \downarrow 0$. Multiplication by $(1-q)^{-\chi_r}$ and integration completes the proof. Boundedness and smoothing justify the differentiations; one may first work on a compact subinterval of $(0, 1)$ and then let its lower endpoint tend to zero. $\square$

We now specialize to $J = DR$. For later use, denote the positive integral operator by

$$(\mathcal{K}_a f)(q) = a(1-q)^a \int_0^q \frac{f(t)}{(1-t)^{a+1}} dt.$$

Write

$$M_1(q) = \sum_{j \geq 1} e_j q^{2j-1}, \quad e_j = 2ja_{2j}.$$

For odd $\ell \geq 1$, define

$$H_{1,\ell}(q) = q^\ell, \quad H_{r,\ell}(q) = \chi_r (1-q)^{\chi_r} \int_0^q \frac{H_{r-1,\ell}(t)}{(1-t)^{\chi_r+1}} dt.$$

Positivity of the integral operators gives the degree-specific bounds

$$M_r(q) \leq \sum_{j \geq 1} e_j H_{r,2j-1}(q).$$

For the fourth moment, the first three kernels are particularly simple:

$$\begin{aligned} H_{2,1}(q) &= 3q^2 - 4q^3 + 3q^4 - \frac{6}{5}q^5 + \frac{1}{5}q^6, \\ H_{2,3}(q) &= \frac{3}{2}q^4 - \frac{3}{5}q^5 + \frac{1}{10}q^6, \\ H_{2,5}(q) &= q^6. \end{aligned}$$

For every $\ell \geq 5$ and $r \geq 1$,

$$H_{r,\ell}(q) \leq q^{\ell-5} H_{r,5}(q).$$

This follows inductively by replacing $t^{\ell-5}$ by $q^{\ell-5}$ inside the positive integral. Thus only the modes 1, 3, 5 are needed to control every Hermite degree.

A simpler uniform consequence is

$$M_2(q) \leq \kappa_{\text{flow}}(q) M_1(q), \quad \kappa_{\text{flow}}(q) = \frac{6q - 1 + (1 - q)^6}{5q}.$$

It strictly improves the previous coefficient $\kappa(q) = q(3 - q)/(1 + q)$:

$$\kappa(q) - \kappa_{\text{flow}}(q) = \frac{q^3(1 - q)(q^2 - 4q + 5)}{5(1 + q)} > 0.$$

The degree-specific estimate is stronger still, because higher Hermite components are more strongly damped at the intermediate smoothing times.

8.1. Finite rational entropy bounds. Fix $R \geq 2$ and put $u = s^2/q < 1$. Since $\|F_q\|_{\text{op}} \leq 1$, $M_r(q) \leq M_R(q)$ for $r \geq R$. Hence

$$-\frac{1}{N} \mathbb{E} \log \det(I - uF_q^2) \leq \sum_{r=1}^R \frac{u^r}{r} M_r(q) + \frac{u^{R+1}}{(R+1)(1-u)} M_R(q).$$

Define

$$L_\ell(q, u) = \sum_{r=1}^R \frac{u^r}{r} H_{r,\ell}(q) + \frac{u^{R+1}}{(R+1)(1-u)} H_{R,\ell}(q).$$

The logarithmic cost is at most

$$2a_2 L_1 + 4a_4 L_3 + \sum_{j \geq 3} 2ja_{2j} q^{2j-6} L_5.$$

For $\ell \in \{1, 3, 5\}$, all the functions $H_{r,\ell}$ are polynomials with rational coefficients: the first integration has degree at most $\chi_2 = 6$, and inductively the preceding degree is strictly smaller than χ_r . Thus these bounds are exact rational functions of rational q, u . For $R = 8$, the largest degree is $\chi_8 = 120$.

For exact evaluation, write $y = 1 - q$. The polynomial recursion is particularly short:

$$\mathcal{K}_a[y^k] = \frac{a}{a-k} (y^k - y^a), \quad k < a.$$

All required inputs have degree below the next value of χ_r , so no logarithmic terms occur. This is the recursion used by the verifier.

8.2. A commutator correction to the entropy flow. The fourth-moment flow has a further correction which retains noncommutativity. Continue to write $F_q = P_{\sqrt{q}} J$ and set

$$D_i = \partial_i F_q, \quad \mathcal{C}(q) = \frac{1}{N} \mathbb{E} \sum_i \|[F_q, D_i]\|_{\text{HS}}^2.$$

The exact derivative formula is

$$M_2'(q) = \frac{6}{qN} \mathbb{E} \text{tr} \left(F_q^2 \sum_i D_i^2 \right) - \frac{\mathcal{C}(q)}{q}. \quad (39)$$

Indeed, for symmetric matrices F, D , $\|[F, D]\|_{\text{HS}}^2 = 2 \text{tr}(F^2 D^2) - 2 \text{tr}(F D F D)$. Consequently

$$M_2'(q) \leq \frac{6}{1-q} (M_1(q) - M_2(q)) - \frac{\mathcal{C}(q)}{q}, \quad (40)$$

and hence

$$M_2(q) \leq (\mathcal{K}_6 M_1)(q) - (1-q)^6 \int_0^q \frac{\mathcal{C}(t)}{t(1-t)^6} dt. \quad (41)$$

We next give an explicit lower bound for this correction. Write $J = \sum_{n \geq 1, n \text{ odd}} J_n$ for the matrix Hermite expansion, and put $e_n = N^{-1} \mathbb{E} \|J_n\|_{\text{HS}}^2$. Thus $e_1 = m$ and $\sum_n e_n \leq 1$. With $J_1 = L = \sum_i G_i A_i$, define

$$c = \frac{1}{N} \sum_{\ell, i} \|[A_\ell, A_i]\|_{\text{HS}}^2 = 2(3v - t_4) \geq 0.$$

Then, for every $0 < q < 1$,

$$\mathcal{C}(q) \geq q^2 c - 4 \sum_{n \geq 3, n \text{ odd}} n e_n q^{n+1}. \quad (42)$$

If $c > 0$, a stronger, nonlinear version is

$$\mathcal{C}(q) \geq \frac{1}{c} \left(qc - 2 \sum_{n \geq 3, n \text{ odd}} n e_n q^n \right)_+^2. \quad (43)$$

Here is a proof which also fixes the normalization of the projection. Let $B_{\ell i} = \mathbb{E}[G_\ell[F_q, \partial_i F_q]]$. Gaussian Bessel and antisymmetrization in the two ordered indices give

$$N \mathcal{C}(q) \geq \sum_{\ell, i} \|B_{\ell i}\|_{\text{HS}}^2 \geq \sum_{\ell, i} \left\| \frac{B_{\ell i} - B_{i\ell}}{2} \right\|_{\text{HS}}^2.$$

Gaussian integration by parts cancels the second derivatives and gives

$$\frac{B_{\ell i} - B_{i\ell}}{2} = \mathbb{E}[\partial_\ell F_q, \partial_i F_q] = \sum_{n \geq 1, n \text{ odd}} q^n R_{n, \ell i}, \quad R_{n, \ell i} = \mathbb{E}[\partial_\ell J_n, \partial_i J_n].$$

The last equality uses orthogonality of distinct differentiated Hermite degrees. In particular $R_{1, \ell i} = [A_\ell, A_i]$.

For any symmetric matrices U_i , a trace expansion gives

$$\begin{aligned} \sum_{\ell, i} \langle [A_\ell, A_i], [U_\ell, U_i] \rangle_{\text{HS}} &= 2 \left\| \sum_\ell A_\ell U_\ell \right\|_{\text{HS}}^2 - 2 \sum_{\ell, i} \text{tr}(A_i A_\ell U_i U_\ell) \\ &\geq -2 \sum_i \|U_i\|_{\text{HS}}^2. \end{aligned}$$

For the inequality, Cauchy–Schwarz and $\sum_i A_i^2 \leq I$ give

$$\left| \sum_{\ell, i} \text{tr}(A_i A_\ell U_i U_\ell) \right| \leq \left(\sum_{\ell, i} \|A_\ell U_i\|_{\text{HS}}^2 \right)^{1/2} \left(\sum_{\ell, i} \|U_\ell A_i\|_{\text{HS}}^2 \right)^{1/2} \leq \sum_i \|U_i\|_{\text{HS}}^2.$$

Apply this with $U_i = \partial_i J_n$ and take expectations. The derivative identity for Hermite chaoses yields

$$\frac{1}{N} \langle R_1, R_n \rangle \geq -2n e_n.$$

Writing $R(q) = \sum_n q^n R_n$, the elementary inequality $\|R(q)\|^2 \geq 2q \langle R(q), R_1 \rangle - q^2 \|R_1\|^2$ proves (42). Cauchy–Schwarz against R_1 proves (43). All series converge for $q < 1$, since $\sum_n e_n \leq 1$.

For a simpler polynomial consequence, suppose $q^2 \leq 3/5$. The sequence nq^n decreases through odd integers $n \geq 3$, so

$$\mathcal{C}(q) \geq q^2 c - 12q^4(1 - m). \quad (44)$$

Inserting this in (41) gives

$$M_2(q) \leq (\mathcal{K}_6 M_1)(q) - A_2(q)(6v - 2t_4) + 12B_2(q)(1 - m), \quad (45)$$

where the exact rational polynomials are

$$\begin{aligned} A_2(q) &= (1 - q)^6 \int_0^q \frac{t}{(1 - t)^6} dt = \frac{q^2}{2} - q^3 + \frac{3q^4}{4} - \frac{3q^5}{10} + \frac{q^6}{20}, \\ B_2(q) &= (1 - q)^6 \int_0^q \frac{t^3}{(1 - t)^6} dt = \frac{q^4}{4} - \frac{3q^5}{10} + \frac{q^6}{20}. \end{aligned}$$

For every $r \geq 3$ define $A_r = \mathcal{K}_{r(2r-1)} A_{r-1}$ and $B_r = \mathcal{K}_{r(2r-1)} B_{r-1}$. Positivity of the flow propagates (45) to all higher moments, with the same linear combination $-A_r(6v - 2t_4) + 12B_r(1 - m)$. These functions are again rational polynomials.

With entropy truncated at moment R and the same geometric remainder as before, put

$$\mathbf{U} = \sum_{r=2}^R \frac{u^r}{r} A_r + \frac{u^{R+1}}{(R+1)(1-u)} A_R, \quad \mathbf{V} = \sum_{r=2}^R \frac{u^r}{r} B_r + \frac{u^{R+1}}{(R+1)(1-u)} B_R.$$

The doubled entropy cost improves by the explicit correction $-\mathbf{U}(6v - 2t_4) + 12\mathbf{V}(1 - m)$. Since $v \geq m^2$, for any fixed m_0 it is at most

$$2\mathbf{U}t_4 - 12(\mathbf{U}m_0 + \mathbf{V})m + 6\mathbf{U}m_0^2 + 12\mathbf{V}.$$

This can be combined with a scalar moment certificate by retaining its t_4 term. Section 10 uses this correction in the exact numerical certificate.

Finally, the leading small- q term records the initial noncommutativity:

$$\mathcal{C}(q) = q^2 c + O(q^3) = 2q^2(3v - t_4) + O(q^3).$$

For bounded J this follows by first restricting to sufficiently small q , where the Hermite expansions and their first derivatives converge in L^4 , and then multiplying the expansions $F_q = \sqrt{q} L + O_{L^4}(q^{3/2})$ and $DF_q = \sqrt{q} DL + O_{L^4}(q^{3/2})$.

9. REFINEMENTS OF THE CUBIC HESSIAN PROJECTION

9.1. Even moments of a Gaussian matrix series. For every symmetric Gaussian series with $S \preceq I$, the standard matrix moment argument [13] gives

$$t_{2k} \leq (2k - 1)t_{2k-2}, \quad t_{2j} = N^{-1} \mathbb{E} \operatorname{tr} L^{2j} = \mathbb{E} X^{2j}, \quad k \geq 2. \quad (46)$$

Indeed, Gaussian integration by parts expands Nt_{2k} as

$$\sum_i \sum_{r=0}^{2k-2} \mathbb{E} \operatorname{tr}(A_i L^r A_i L^{2k-2-r}).$$

In an eigenbasis of L , weighted arithmetic-geometric mean, followed by summation over the two matrix indices, bounds each term by $\mathbb{E} \operatorname{tr}(A_i^2 |L|^{2k-2})$. Summing over i proves (46). These inequalities do not use full symmetry of T and do not specify which Hermite degrees account for the failure of L to be a contraction. The new bound uses precisely that missing information.

9.2. A marginal refinement.

Lemma 9.1. *Let $Z = N^{-1}\|\Pi_3(L^3)_3\|_{L^2}^2$ and $t = N^{-1}\mathbb{E} \operatorname{tr} L^4$. For every fixed $0 < \lambda < 1$, put*

$$C_\lambda = 1 + (1 - \lambda)^2, \quad \Delta_\lambda = t - 3\lambda m + \frac{3}{2}\lambda^2.$$

Then $\Delta_\lambda \geq 0$ and

$$Z \leq \frac{4\Delta_\lambda}{C_\lambda}.$$

Proof. Retain the quartic tensor H used in the preceding projection argument, and let $\mathcal{Q}_{ij} = \sum_{b,c,d} H_{ibcd} H_{jbcd}$. Put $\Phi(A) = \sum_i A_i A A_i$. The three terms of $H(x, \cdot, \cdot, \cdot)$ each have squared norm $x^\top \Phi(S)x$. Hence Cauchy–Schwarz gives

$$0 \preceq \mathcal{Q} \preceq 9\Phi(S) \preceq 9S, \quad \operatorname{tr} \mathcal{Q} = 6Nt - 9 \operatorname{tr} S^2.$$

The same grafting argument gives $Z \leq 2 \operatorname{tr}(S\mathcal{Q})/(3N)$. For $0 \leq s \leq 1$,

$$(s - \lambda)^2 - s(C_\lambda s - 1) = (1 - s)[\lambda^2 + (1 - \lambda)^2 s] \geq 0,$$

so $s(C_\lambda s - 1)_+ \leq (s - \lambda)^2$. Consequently

$$\begin{aligned} C_\lambda \operatorname{tr}(S\mathcal{Q}) &\leq \operatorname{tr} \mathcal{Q} + \operatorname{tr}((C_\lambda S - I)_+ \mathcal{Q}) \\ &\leq \operatorname{tr} \mathcal{Q} + 9 \operatorname{tr}(S(C_\lambda S - I)_+) \\ &\leq \operatorname{tr} \mathcal{Q} + 9 \operatorname{tr}(S - \lambda I)^2 = 6N\Delta_\lambda. \end{aligned}$$

This proves the bound for Z . Finally, $\Delta_\lambda = t - 3m^2/2 + 3(m - \lambda)^2/2 \geq 0$. $\square$

9.3. Retaining the sixth matrix moment. We now retain a sixth moment instead of eliminating it. Define

$$r_1 = \sum_{i,j} \operatorname{tr}(S A_i A_j A_i A_j), \quad r_2 = \sum_i \operatorname{tr}(A_i S A_i S), \quad \Phi(B) = \sum_i A_i B A_i,$$

$$R_0 = \sum_i \|\Phi(A_i)\|_{\text{HS}}^2, \quad K_0 = \sum_{i,j,k} \operatorname{tr}(A_i A_j A_k A_i A_j A_k),$$

and write

$$m = \frac{1}{N} \operatorname{tr} S, \quad v = \frac{1}{N} \operatorname{tr} S^2, \quad w = \frac{1}{N} \operatorname{tr} S^3, \quad u = \frac{r_2}{N}, \quad z = \frac{r_1}{N}, \quad d = \frac{R_0}{N}, \quad k = \frac{K_0}{N}.$$

Let $t_6 = N^{-1}\mathbb{E} \operatorname{tr} L^6$ and $Z = N^{-1}\|\Pi_3(L^3)_3\|_{L^2}^2$. Then

$$\boxed{Z \leq \frac{8t_6 - 18m^3}{17}.} \tag{47}$$

In fact the proof gives the stronger form

$$Z \leq \frac{8t_6 - 18v^2/m}{17} \quad (m > 0). \tag{48}$$

The additional Gram inequality below is the step that retains useful sixth-moment information.

Contraction identities. Averaging over the fifteen distinct external-index arrangements of the chain tensor, and applying Wick's formula, gives

$$Z = \frac{2u + 8z + 16d + 4k}{5}, \quad t_6 = 2w + 3u + 6z + 3d + k.$$

Each row of the chain Gram matrix contains one entry r_2 , four entries r_1 , eight entries R_0 , and two entries K_0 ; the third-chaos isometry contributes the factor $3!$. The earlier grafting estimate, before discarding its marginal, is $Z \leq 2 \operatorname{tr}(SQ)/(3N) = 2u + 4z$. Thus

$$G_0 := 2u + 3z - 4d - k \geq 0.$$

There is also a five-index Gram inequality

$$E_0 := u - z - 2d + 2k \geq 0.$$

For completeness, with $U_{ab|c|de} = \sum_{p,q} T_{abp} T_{pcq} T_{qde}$, put

$$\begin{aligned} F_a &= U_{bc|a|de} + U_{bd|a|ce} + U_{be|a|cd}, \\ F_b &= U_{ac|b|de} + U_{ad|b|ce} + U_{ae|b|cd}. \end{aligned}$$

Direct contraction gives $\|F_a\|^2 = \|F_b\|^2 = 3r_2 + 6K_0$ and $\langle F_a, F_b \rangle = 3r_1 + 6R_0$. Hence

$$\|F_a - F_b\|^2 = 6(r_2 - r_1 - 2R_0 + 2K_0) = 6NE_0.$$

Consequently

$$8t_6 - 17Z = 16w + 2u + \frac{36}{5}G_0 + \frac{4}{5}E_0 \geq 16w + 2u. \quad (49)$$

The remaining estimate is an elementary use of full symmetry. Diagonalize S , with eigenvalues s_i , and, when $m > 0$, give (i, j, k) probability $T_{ijk}^2/(Nm)$. Its three coordinates are exchangeable and

$$\mathbb{E}s_i = v/m, \quad \mathbb{E}s_i^2 = w/m, \quad \mathbb{E}(s_i s_j) = u/m.$$

The variance of $s_i + s_j + s_k$ gives $w + 2u \geq 3v^2/m$. Also $w \geq v^2/m$ by the scalar moment inequality for the eigenvalues of S . Hence

$$16w + 2u \geq 15w + 3v^2/m \geq 18v^2/m \geq 18m^3.$$

Together with (49), this proves (47); the case $m = 0$ is trivial.

10. A SEPTIC TEST AND THE FINAL CERTIFICATE

Several prescribed Gaussian chaoses can be tested together without computing a full tensor Gram matrix. Let Q be an odd polynomial of degree at most seven, and let $r \in \mathbb{R}$. Then

$$4a_4 + 6a_6 + 8a_8 \geq \mathbb{E} \left[2(XQ(X) - |Q(X)|) - (Q(X) - rX)^2 \right]. \quad (50)$$

Indeed, contraction of J and chaos orthogonality give

$$(\mathbb{E}[XQ(X) - |Q(X)|])_+ \leq \sqrt{(4a_4 + 6a_6 + 8a_8)V}, \quad V = \mathbb{E}(Q(X) - rX)^2.$$

For the last estimate, retain the third, fifth, and seventh chaos components of $Q(L)$ and bound their total squared norm by that of $Q(L) - rL$. Completing a square, $2\sqrt{(4a_4 + 6a_6 + 8a_8)V} - V \leq 4a_4 + 6a_6 + 8a_8$, proves (50). This proof applies to any real r ; selecting r near the first-chaos regression coefficient is useful numerically.

10.1. The rational certificate. This section gives a complete rational certificate combining the spectral matrix-budget inequality, the sixth-moment Hessian projection, the septic test, and the commutator correction to the entropy flow. Write $A_4 = 4a_4$ and define

$$d_b = 16c_b - 6 - 4\lambda_b, \quad \alpha_b = \frac{6d_b}{(6A_0 + d_b)^2}, \quad \beta_b = \frac{36A_0^2}{(6A_0 + d_b)^2},$$

$$F_4 = \frac{16c_b^2 - 4\lambda_b^2}{3}, \quad F_2 = 2 + 2\lambda_b^2 - 8c_b^2.$$

The parameters below satisfy $d_b > 0$ and $F_2 < 0$. Thus the preceding matrix-budget lemma gives

$$\alpha_b A_4 + \beta_b \geq 2\gamma(t_4 - \mathbb{E}X^3) - \gamma^2(F_4 t_4 + F_2 m^2 - 4\lambda_b m + 2\lambda_b^2).$$

Blend this with weight $1 - \eta$ and the sixth-moment projection inequality with weight η , using the same γ in both. Add ϑ times the septic Hessian test with

$$Q(x) = b_q x^3 (c_q^2 - x^2)^2.$$

Introduce a positive coefficient ξ , which will absorb the positive t_4 term from the entropy correction. Set

$$k_4 = 2\gamma - \gamma^2(1 - \eta)F_4 - \xi, \quad k_3 = -2\gamma, \quad k_6 = -8\gamma^2\eta/17,$$

and

$$\begin{aligned} \mathcal{P}(x) = & k_6 x^6 + k_4 x^4 + k_3 x^3 + \vartheta \{2(x-1)Q(x) - (Q(x) - r_q x)^2\} - c_0 - c_1 x - c_2 x^2 \\ & + \theta_6 \left(\frac{x^{12}}{10395} - \frac{x^{10}}{945} \right) + \theta_{10} \left(\frac{x^{20}}{654729075} - \frac{x^{18}}{34459425} \right). \end{aligned} \quad (51)$$

All the following terminating decimals are exact rational numbers:

γ	0.46943966493022	η	0.03147922506626
λ_b	0.83422581962954	c_b	0.88617587397029
A_0	0.11733062741006	m_0	0.84031353613959
c_0	-0.03861305150538	c_1	0.29647789431113
c_2	-0.96425074771268	ϑ	0.16218889900094
θ_6	0.01246697666460	θ_{10}	0.00826314934702
b_q	0.00363340143362	c_q	3.75487290159014
r_q	0.77810686755460	ξ	0.00322745373409

Exact Sturm arithmetic applied to $\mathcal{P} - 10^{-6}$ gives ten variations at both 0 and $+\infty$, and its constant and leading coefficients are positive. In particular

$$\mathcal{P}(x) > 10^{-6} \quad (x \geq 0).$$

The scalar moment recursions and $\mathbb{E}X \geq m$, together with $c_1 > 0$, therefore prove

$$4k_* a_4 + 6\vartheta a_6 + 8\vartheta a_8 \geq P_* m - Q_* + \xi t_4, \quad k_* = (1 - \eta)\alpha_b + \eta + \vartheta. \quad (52)$$

For completeness, the exact definitions of the constants are

$$\begin{aligned} f_0 &= c_0 - 2\gamma^2(1 - \eta)\lambda_b^2 - (1 - \eta)\beta_b, \\ f_1 &= c_1 + c_2 + 4\gamma^2(1 - \eta)\lambda_b, \\ f_2 &= -\gamma^2(1 - \eta)F_2, \quad f_3 = 18\gamma^2\eta/17, \\ P_* &= f_1 + 2f_2 m_0 + 3f_3 m_0^2, \quad Q_* = f_2 m_0^2 + 2f_3 m_0^3 - f_0. \end{aligned}$$

For $m > 0$, before taking the tangent at m_0 , the right side of (52) is $f_0 + f_1 m + f_2 m^2 + f_3 v^2/m + \xi t_4$. Use $v \geq m^2$ and then the convexity of $f_0 + f_1 m + f_2 m^2 + f_3 m^3$, since $f_2, f_3 \geq 0$. The case $m = 0$ follows directly or by continuity. Numerically,

$$k_* = 1.1084841508965669\dots, \quad P_* = 1.0968808948996438\dots, \quad Q_* = 0.7956615808018579\dots$$

Use the following exact transport parameters:

$$\begin{array}{c|c} \beta & 0.7866529541 \\ s & 0.1255274413 \\ \zeta & 0.6774479657 \\ \omega & 0.0128557668 \end{array} \quad \begin{array}{c|c} \rho & 0.8456140263 \\ \sigma & 0.6908074308 \\ M & 1.3417650561 \end{array}$$

Put $q = \sigma^2$, $u = s^2/q$ and use the flow kernels through order $r = 8$. In particular $q^2 < 3/5$, as required for the polynomial commutator bound. To avoid confusion with the geometric transport energies, denote its correction coefficients by U, V :

$$U = \sum_{r=2}^8 \frac{u^r}{r} A_r(q) + \frac{u^9 A_8(q)}{9(1-u)}, \quad V = \sum_{r=2}^8 \frac{u^r}{r} B_r(q) + \frac{u^9 B_8(q)}{9(1-u)}.$$

They are exact rational numbers, with approximate values

$$U \simeq 2.07356963 \cdot 10^{-5}, \quad V \simeq 3.40224854 \cdot 10^{-6}.$$

With the kernels L_ℓ of Section 8, define explicitly

$$d_j = \frac{b_j}{M} + \begin{cases} 2L_1, & j = 1, \\ 4L_3, & j = 2, \\ 2jq^{2j-6}L_5, & j \geq 3, \end{cases}$$

where b_j is given by (24). The exact inequalities required for the dual are

$$\begin{aligned} d_1 &\leq 2\omega P_* + 24(Um_0 + V), \\ d_2 &\leq -4\omega k_*, \quad d_3 \leq -6\omega \vartheta, \quad d_4 \leq -8\omega \vartheta, \quad d_j \leq 0 \quad (j \geq 5), \\ 2U &\leq \omega \xi. \end{aligned}$$

The final inequality absorbs $2Ut_4$ using (52). The term linear in m is absorbed in the first dual inequality. Therefore the squared exponential base is at most

$$C_*^2 = M \exp \left(\frac{M_0}{M} - 1 + \omega Q_* + 6Um_0^2 + 12V \right), \quad M_0 = 1 + \frac{1 + \zeta^2}{2\zeta} (1 - \beta\rho)^2 + \frac{1 + \zeta^2}{4} \beta^2. \quad (53)$$

This is an exact expression in the displayed rational parameters and the explicitly defined rational flow kernels. Its value is

$$C_* = 1.1663732215981908779873767633 \dots < 1.166374.$$

Consequently the improved result is

$$\boxed{\limsup_{n \rightarrow \infty} b(n)^{1/n} \leq C_* < 1.166374.}$$

The accompanying script `verify_borsuk_hessian.py` is a self-contained, standard-library certificate. It checks the positive scalar polynomial by exact Sturm arithmetic, evaluates the stated flow and commutator recursions, and checks the numerical spectral inequalities. These computations do not verify the analytic derivation of those inequalities. The first two spectral margins exceed 1.98×10^{-8} and 2.01×10^{-8} ; the commutator-absorption margin exceeds 1.99×10^{-8} . The margins for degrees 3, 4 and $5 \leq j \leq 50$ exceed 0.0963, 0.1203, and 0.1519. The decreasing infinite-tail envelope has margin greater than 0.2257. A rational Taylor upper bound for the exponential leaves squared-base margin greater than 1.81×10^{-6} below $(1.166374)^2$. The numerical search preceding the conservative polynomial slack gave $1.16637312697 \dots$; that unrounded value is not used in the proof.

11. HIGHER MOMENTS AND A HIERARCHY OF HESSIAN TESTS

11.1. The all-order tensor-budget pattern. The factor $3/4$ in Theorem 4.1 is the first member of an all-order sequence. Fix an even integer $p \geq 4$, and write

$$J_{p-1} = I_{p-1}(D_p), \quad A_p = \frac{(p-1)!}{N} \|D_p\|^2 = p a_p.$$

The tensor D_p is fully symmetric in its $p+1$ indices. Let H_p be any deterministic fully symmetric tensor of order p , and attach the first-chaos tensor T at one index:

$$(V_p)_{i_1 \dots i_{p-1} ab} = \sum_k (H_p)_{i_1 \dots i_{p-1} k} T_{kab}.$$

Put

$$q_p = \frac{\|H_p\|^2}{N}, \quad r_p = \frac{\langle H_p, S_1 H_p \rangle}{N}, \quad \ell_p = \frac{\sqrt{(p-1)!}}{N} |\langle D_p, V_p \rangle|.$$

Here S_1 acts on one tensor index. Then

$$\ell_p^2 \leq A_p r_p, \quad \ell_p^2 \leq \frac{p}{p-1} r_p \left(1 - \frac{r_p}{q_p}\right). \quad (54)$$

When $q_p = 0$, both sides vanish. Equivalently,

$$\ell_p^2 \leq q_p \begin{cases} A_p \left(1 - \frac{p-1}{p} A_p\right), & 0 \leq A_p \leq \frac{p}{2(p-1)}, \\ \frac{p}{4(p-1)}, & A_p \geq \frac{p}{2(p-1)}. \end{cases} \quad (55)$$

Thus the nonlinear penalty coefficients are $3/4, 5/6, 7/8, \dots$ for $p = 4, 6, 8, \dots$

For completeness, let $B_{p-1} = \mathbb{E} J_{p-1}^2$. Orthogonality gives $S + B_{p-1} \leq I$, and flattening any index of D_p gives $D_p D_p^* = B_{p-1} / (p-1)!$. Ordinary tensor Cauchy–Schwarz gives the first bound in (54). Applying the matrix budget at one of the $p-1$ external indices of H_p , as in (11), gives

$$\ell_p^2 \leq \frac{1}{N} \langle H_p, (I - S)_1 S_p H_p \rangle = r_p - c_p, \quad c_p = \frac{\langle H_p, S_1 S_2 H_p \rangle}{N}.$$

Writing $b_p = \langle H_p, S_1^2 H_p \rangle / N \leq r_p$, full symmetry gives

$$p b_p + p(p-1) c_p = \frac{1}{N} \left\| \sum_{i=1}^p S_i H_p \right\|^2 \geq \frac{p^2 r_p^2}{q_p}.$$

This proves the second bound in (54); maximizing the minimum of the two bounds over $r_p \in [0, q_p]$ proves (55).

For $p = 4$, take $H_4 = H$. The cubic-test gap satisfies $g^2 \leq (2/3) \ell_4^2$ and $q_4 = 6\delta$, recovering the nonlinear fourth-chaos bound. At larger orders, (55) controls each grafted fully symmetric test tensor. Applying it to an odd matrix power still requires retaining that power's lower Hermite components; the hierarchy by itself does not eliminate those components or certify a further numerical improvement.

11.2. Tests of arbitrary odd degree. For $j \geq 2$, let $\hat{\Pi}_j$ denote projection of the $(2j-1)$ st matrix chaos onto the fully symmetric $(2j+1)$ -index tensors. For an odd real polynomial P_0 of degree at most $2q-1$, define

$$z_j(P_0) = \frac{1}{N} \|\hat{\Pi}_j(P_0(L))\|_{L^2}^2, \quad 2 \leq j \leq q.$$

The letter P_0 here denotes a scalar polynomial, not a cone projection.

Proposition 11.1. *For every such polynomial,*

$$(\mathbb{E}[XP_0(X) - |P_0(X)|])_+ \leq \sum_{j=2}^q \sqrt{2j a_{2j} z_j(P_0)}. \quad (56)$$

For the highest chaos of the monomial test, one has the uniform bound

$$z_q(x^{2q-1}) \leq \frac{2}{3}(2q-1)!\delta \quad (q \geq 2). \quad (57)$$

Proof. Exact chaos orthogonality and the Hessian property give

$$\frac{1}{N} \mathbb{E} \operatorname{tr}(JP_0(L)) = \mathbb{E}[XP_0(X)] + \frac{1}{N} \sum_{j=2}^q \langle J_{2j-1}, \hat{\Pi}_j(P_0(L))_{2j-1} \rangle.$$

The first term uses $J_1 = L$ and the evenness of $\lambda P_0(\lambda)$. Contraction bounds the left side by $\mathbb{E}|P_0(X)|$; Cauchy–Schwarz proves (56).

For (57), the coefficient of the highest chaos of L^{2q-1} is a path of $2q-1$ copies of T , with $2q+1$ external indices. Start with H and attach $2q-3$ copies of T successively to one external leg, continuing along the new path. Each attachment is a contraction in Euclidean tensor norm, since its flattening has squared operator norm $\|S\|_{\text{op}} \leq 1$. After full symmetrization, the three terms of H produce the same path tensor. Thus the norm of the fully symmetric path is at most $\|H\|/3$. The Hermite isometry and (5) give

$$z_q(x^{2q-1}) \leq \frac{(2q-1)!}{9N} \|H\|^2 = \frac{2}{3}(2q-1)!\delta. \quad \square$$

The same proof also retains the marginal refinement: after the first attachment the squared norm is $\operatorname{tr}(SQ)$, and subsequent attachments are contractions. Therefore, with $\Delta_\lambda = t - 3\lambda m + 3\lambda^2/2$,

$$z_q(x^{2q-1}) \leq \frac{2(2q-1)!}{3} \frac{\Delta_\lambda}{1 + (1-\lambda)^2} \quad (0 < \lambda < 1).$$

The first values of the coefficient in (57) are 4, 80, 3360 for cubic, quintic, and septic tests. For a quintic test, (56) reads

$$(t_6 - \mathbb{E}X^5)_+ \leq \sqrt{4a_4 z_2(x^5)} + \sqrt{6a_6 z_3(x^5)}, \quad z_3(x^5) \leq 80\delta. \quad (58)$$

A separate estimate for $z_2(x^5)$ would distinguish the fourth and sixth Hermite energies. The joint test of Section 10 gives another route: it controls their partial sum, and the septic version used in the numerical bound includes the eighth Hermite energy as well. The leading-chaos estimate by itself does not provide this information.

There are two systematic extensions. First, for positive weights η_j , the square of the left side of (56) is at most

$$\left(\sum_{j=2}^q 2j\eta_j a_{2j} \right) \left(\sum_{j=2}^q \frac{z_j(P_0)}{\eta_j} \right).$$

Second, testing the smoothed contraction $P_\tau J$ replaces the left side by $(\mathbb{E}[\tau XP_0(X) - |P_0(X)|])_+$ and replaces a_{2j} on the right by $\tau^{4j-2} a_{2j}$. Both versions remain exact.

At each fixed degree the projected norms are finite sums of contractions of copies of T . Gram matrices of their external-index permutations are positive semidefinite. Wick pairings express the scalar even moments in contractions of the same type. Consequently one can retain scalar moment positivity, tensor Gram positivity, and the prescribed Hermite degrees in one finite algebraic relaxation. Appendix A supplies an explicit sixth-moment instance. The hierarchy is rigorous at every level; no completeness assertion is made. The numerical bound uses the cubic projection and

matrix-budget constraints together with one joint septic test and the commutator correction to the entropy flow.

11.3. Retaining the covariance matrix in higher moment bounds. For a symmetric Gaussian matrix series $L = \sum_i G_i A_i$ with $S = \sum_i A_i^2 \preceq I$, the even moments satisfy the Loewner bound

$$\mathbb{E}L^{2q} \preceq (2q-1)!! S. \quad (59)$$

One proof applies Wick's formula to each paired word. Define the contraction $Vx = \sum_i A_i x \otimes e_i$, so $V^*V = S \preceq I$. Read a paired word from right to left, applying V when a label first appears and V^* when that label appears again. Tensor-factor permutations bring the required open label next to the physical factor. All intermediate maps are contractions. The first and last operations factor the paired word as V^*CV , with $\|C\| \leq 1$. Consequently its quadratic form has absolute value at most $x^\top Sx$. Summing the $(2q-1)!!$ pairings proves (59).

This matrix information can be coupled to the projection bound $Z \leq (8t_6 - 18v^2/m)/17$, rather than replacing v immediately by m^2 . Here is one explicit affine family, suited to scalar polynomial certificates. For any $\lambda \geq 0$ and $0 < \eta < 1$,

$$\begin{aligned} Z &\leq \frac{8}{17}t_6 - \frac{54\lambda^2}{17}m + \frac{36\lambda^3}{17} \\ &\quad + \frac{144\lambda^2\eta}{17 \cdot 945}t_{10} - \frac{144\lambda\eta(1-\eta)}{187 \cdot 945}t_{12}. \end{aligned} \quad (60)$$

To prove this, let $A = \mathbb{E}L^{10}$, $b = N^{-1} \text{tr}(SA)$, and $e_\lambda = N^{-1} \text{tr}(S - \lambda I)^2 = v - 2\lambda m + \lambda^2$. By (59), $0 \preceq A \preceq 945S$, and hence $N^{-1} \text{tr} A^2 \leq 945b$. Therefore

$$b \leq \lambda t_{10} + \sqrt{945e_\lambda b} \leq \lambda t_{10} + \eta b + \frac{945e_\lambda}{4\eta}.$$

Gaussian integration by parts and the same spectral Young inequality used for scalar even moments give $t_{12} \leq 11b$. It follows that

$$v \geq 2\lambda m - \lambda^2 + \frac{4\eta}{945} \left(\frac{1-\eta}{11} t_{12} - \lambda t_{10} \right).$$

Finally $v^2/m \geq 2\lambda v - \lambda^2 m$; insert this estimate in the projection bound, and then insert the displayed lower bound for v . The coefficient of v is nonpositive because $\lambda \geq 0$. This gives (60). Again $m = 0$ is trivial.

The same argument works with $A = \mathbb{E}L^{2q}$ and $(2q-1)!!$ in place of 945. The next scalar moment then obeys $t_{2q+2} \leq (2q+1)N^{-1} \text{tr}(SA)$.

12. ADDITIONAL GEOMETRIC CONSTRAINTS

The following restrictions can be used in the same transport framework. They sharpen particular configurations, but are not required for Theorem 1.1.

12.1. Several common latitudes. Let $E \subset \mathbb{R}^N$ contain μ , with $\dim E = k = o(N)$, and suppose that the vertices in a part have the same projection c_E onto E . Put $q_E = |c_E|^2$, $w = 1 - q_E$, and $a_E = N^{-1/2} \mu \cdot c_E$. If $x = c_E + \sqrt{w}e$, an orthogonal neighbor has the form

$$y = c_E - \frac{q_E}{\sqrt{w}}e + y_0, \quad y_0 \perp E, e, \quad |y_0|^2 = \frac{2w-1}{w}.$$

The shift $z = x - c_E$ still satisfies $z \cdot (x - y) = 1$. Since $\mathcal{H}$ is centered and Lipschitz, $\mathbb{E}|P_E \mathcal{H}|^2 \leq k \text{Lip}(\mathcal{H})^2 = o(N)$. The proof of (19) therefore gives the improved cost

$$1 + s^2 V + \left(\sqrt{(2w-1)Q_0} + \beta a_E \sqrt{w} \right)^2 + o(1). \quad (61)$$

A Euclidean η -net for the unit ball in E creates at most $(1 + 2/\eta)^k = \exp(o(N))$ approximate-projection classes, so this refinement is compatible with the exponential bound.

In particular the radial lift has a common coordinate $\ell = \sqrt{1 - 2r^2}$ along its added axis g . Write $\gamma = f \cdot g$. When $|\gamma| < 1$, removing both f and g gives

$$q_E = \alpha^2 + \frac{(\ell - \gamma\alpha)^2}{1 - \gamma^2}, \quad a_E = \sqrt{h}\alpha. \quad (62)$$

The old estimate removes only α^2 . Thus the added axis can strictly improve the support bound. If $q_E > 1/2$, there are no orthogonal edges in the part. When $q_E \leq 1/2$, the cost in (61) decreases as q_E increases with a_E fixed. A universal further improvement would require information preventing $\ell = \gamma\alpha$ in the configurations governing the maximum.

12.2. Neighbors lie in an exposed face. For $x \in D \cap \mathbb{S}^{N-1}$, let $F_x = D \cap x^\perp$ and write $\text{sdim}(C) = \mathbb{E}|P_C G|^2$ for statistical dimension. Then

$$\text{sdim}(F_x) \leq \frac{N}{2} - \frac{(\mu \cdot x)^2}{4}. \quad (63)$$

To prove this, put $p_D = P_D G$ and $v_F = P_{F_x} G$. Projection optimality gives

$$|p_D|^2 - |v_F|^2 - |p_D - v_F|^2 = 2v_F \cdot (p_D - G) \geq 0.$$

Since $v_F \cdot x = 0$, $|p_D - v_F|^2 \geq (p_D \cdot x)^2$. Now use $\mathbb{E}|p_D|^2 = N/2$, $\mathbb{E}p_D = \mu/2$, and Jensen. This is a genuine restriction on the whole neighbor set, beyond its residual spherical radius. The field $\mathcal{H}(G)$ is generally non-Gaussian, so (63) cannot simply replace Q_0 in (19); a joint estimate would be needed.

12.3. Why covariance and face width cannot be combined automatically. The support field in the transport is generally non-Gaussian. Even isotropic covariance is insufficient to substitute a Gaussian width bound. For example, take $D = \mathbb{R}_+^N$ and

$$R(G) = (|G_1|, \dots, |G_N|), \quad H = R(G) - c\mathbf{1}, \quad c = \sqrt{2/\pi}.$$

This is exactly the field $\mathcal{H}$ at $s = 0$, $\beta = \rho = 1$. It is centered and has covariance $(1 - 2/\pi)I$. For the exposed face $F = D \cap e_1^\perp$, one has $\text{sdim}(F) = (N - 1)/2$, while the law of large numbers gives

$$\frac{1}{N} \left(\sup_{y \in F \cap B_2^N} H \cdot y \right)^2 \longrightarrow q_+ := \mathbb{E}(|Z| - c)_+^2 = 2[(1 + c^2)\overline{\Phi}(c) - c\varphi(c)],$$

where Z is standard normal and $\varphi, \overline{\Phi}$ are its density and upper tail. The same limit holds with the unit sphere in place of the unit ball, since some coordinate of H is positive with probability tending to one. Numerically,

$$q_+ = 0.232398285 \dots > \frac{1 - 2/\pi}{2} = 0.181690113 \dots$$

The strict inequality is also elementary. The centered random variable $V = |Z| - c$ has density $g(v) = 2\varphi(v + c)$ on $[-c, \infty)$ and zero elsewhere. For $0 < t < c$, $g(t) - g(-t) < 0$, whereas for $t > c$ this difference is positive. Since $\mathbb{E}V = 0$,

$$\mathbb{E}[V|V|] = \int_0^\infty t(t - c)(g(t) - g(-t)) dt > 0.$$

Thus $\mathbb{E}(V_+)^2 > \mathbb{E}V^2/2$. A Gaussian field with the same covariance would give the smaller limit $(1 - 2/\pi)/2$ on this face. Hence replacing the support estimate by covariance times statistical dimension would be false even within the transport family itself. A useful face refinement must control the joint distribution of the field and the face.

12.4. Reflection identities and their limitations. Writing $u = p - q$ and $R(u) = p + q$ in Moreau's decomposition shows

$$R \circ R = R, \quad R(au + bR(u)) = aR(u) + bu \quad (a \geq |b|).$$

The latter identity follows because $(a + b)p$ and $(a - b)q$ remain orthogonal vectors of D . At differentiability points,

$$J(u)u = R(u), \quad J(u)R(u) = u, \quad J(u)^2 u = u.$$

The first equality is Euler's identity, and the second follows by differentiating $|R(u)|^2 = |u|^2$. In terms of the cubic coefficients one also has

$$\operatorname{tr} A_i = \mu_i, \quad \mathbb{E}(\operatorname{tr} L)^2 = |\mu|^2.$$

Indeed $\mu_j = \sum_i \mathbb{E}[G_i J_{ji}] = \sum_i (A_i)_{ji} = \sum_i (A_j)_{ii}$. These identities are additional geometric data, but their immediate trace-norm consequences lose a factor of N and do not improve the present exponential estimate.

13. FURTHER GEOMETRIC REFINEMENTS

13.1. Using the original acute cone: a measurable norm defect. The self-dual completion is convenient but can discard a quantitative advantage. Let $C \subset \mathbb{R}^N$ be a closed convex acute cone, so $C \subset C^*$, and define

$$E_C(x) = P_C x + P_C(-x), \quad e_C = N^{-1} \mathbb{E}|E_C(G)|^2, \quad \delta(C) = \mathbb{E}|P_C G|^2.$$

Here duals use the nonnegative inner-product convention. The map E_C is even, positively homogeneous, and a gradient. Indeed, it is the gradient of $\frac{1}{2}|P_C x|^2 - \frac{1}{2}|P_C(-x)|^2$. Almost everywhere its derivative is $DP_C(x) - DP_C(-x)$. Both matrices are symmetric and lie between 0 and I , so their difference lies between $-I$ and I . It follows that E_C is 1-Lipschitz and $|E_C(x)| \leq |x|$. Moreau decomposition also gives the pointwise identities

$$E_C(x) + x = P_C x + P_{C^*} x \in C^*, \quad E_C(x) - x = P_C(-x) + P_{C^*}(-x) \in C^*.$$

Thus, for vertices in C , the smoothed compensator $P_\rho E_C(x) - \rho x$ lies in C^* and has exactly the sign needed in the transport argument. Its mean is $\mu_C = 2\mathbb{E}P_C G \in C$.

The norm defect admits a geometric lower bound:

$$e_C \leq 4 \frac{\delta(C)}{N} \left(1 - \frac{\delta(C)}{N}\right), \quad 1 - e_C \geq \left(1 - 2 \frac{\delta(C)}{N}\right)^2. \quad (64)$$

To prove this, put $O_C(x) = P_C x - P_C(-x)$. The parallelogram identity gives $\mathbb{E}|E_C(G)|^2 + \mathbb{E}|O_C(G)|^2 = 4\delta(C)$. Moreover $\mathbb{E}\langle G, O_C(G) \rangle = 2\delta(C)$, since $\langle x, P_C x \rangle = |P_C x|^2$. Cauchy-Schwarz therefore yields $\mathbb{E}|O_C(G)|^2 \geq 4\delta(C)^2/N$, proving the claim. Since $C \subset C^*$, monotonicity of statistical dimension and $\delta(C) + \delta(C^*) = N$ imply $\delta(C) \leq N/2$. There is a matrix refinement. With $M_C = \mathbb{E}DP_C(G)$, Gaussian integration by parts gives $\operatorname{tr} M_C = \delta(C)$ and the first Hermite projection of O_C is $2M_C G$. Hence

$$e_C \leq \frac{4}{N}(\delta(C) - \operatorname{tr} M_C^2).$$

Write $E_C = \mu_C + \sum_{j \geq 1} (E_C)_{2j}$ and $a_{2j} = N^{-1} \|(E_C)_{2j}\|_2^2$, $h = |\mu_C|^2/N$. The energy identity becomes

$$h + \sum_{j \geq 1} a_{2j} = e_C \leq 1.$$

The bounds derived solely from an even homogeneous gradient with contractive derivative remain valid: in particular the differentiated energy bound, full symmetry of the cubic coefficient tensor,

and the polynomial matrix inequalities used to constrain the higher Hermite energies. Norm preservation must not be silently retained; any use of $h \geq 1/2$ must instead be checked or replaced by $h \geq e_C - 1/2$.

For example, with the uniform-latitude support majorant $A = (1 + \zeta^2)/(2\zeta)$, $B = (1 + \zeta^2)/4$, the second-moment constant becomes

$$1 + A(1 - \beta\rho)^2 + B\beta^2 e_C + \sum_{j \geq 1} \left[s^2 \sigma^{4j-4} + A(\beta\rho^{2j} - s\sigma^{2j-2})^2 - B\beta^2 \right] a_{2j}.$$

Thus any otherwise valid scalar dual certificate gains $B\beta^2(1 - e_C)/M$ in the upper bound for twice the logarithmic exponent, where M is its logarithmic linearization parameter. In particular a statistical-dimension gap $\delta(C) \leq (1/2 - \varepsilon)N$ gives a gain at least $4B\beta^2\varepsilon^2/M$ there. This is a conditional geometric improvement, not a uniform one: self-dual cones have $e_C = 1$. The displayed transport parameters of the 1.166374 certificate satisfy, by direct rational arithmetic, $B\beta^2/(2M) > 840/10000$. Its active moment dual does not use $h \geq 1/2$, so the resulting conditional base is at most

$$1.166374 \exp\{-0.0840(1 - e_C)\} \leq 1.166374 \exp\{-0.3360\varepsilon^2\} \quad \text{if } \delta(C)/N \leq 1/2 - \varepsilon.$$

For this conclusion the statistical-dimension hypothesis is imposed on the original acute cone of each spherical part; no such gap is asserted for arbitrary diameter graphs.

13.2. Selecting among correlated transport proposals. The following entropy observation permits the use of the mutual acuteness of all neighbors, rather than their individual norm alone. Let $J = DR$, $F = P_\sigma J$, and $\mathcal{T}(x) = x - s(P_\sigma R(x) - \mu)/\sigma^2$, with $0 < s < \sigma < 1$. Put $t = s/\sigma$ and $\kappa = \sigma^2/(1 - \sigma^2)$. For $D_i = \partial_i F$, conditional Gaussian Bessel inequality gives

$$\sum_i D_i^2 \preceq \kappa(I - F^2) \preceq \kappa I.$$

The tensor $(D_i)_{ab}$ is fully symmetric. Consequently the adjoint of the linear map $z \mapsto \sum_i z_i D_i$ has norm at most $\sqrt{\kappa}$: its Gram matrix is $\sum_i D_i^2$. Differentiating the logarithmic determinant therefore gives

$$|\nabla \log \det D\mathcal{T}|^2 \leq \kappa t^2 \|(I - tF)^{-1}\|_{\text{HS}}^2 \leq \frac{\kappa t^2}{(1 - t)^2} N. \quad (65)$$

In particular its variance is $O(N)$.

Let $G_1, \dots, G_k$ have standard Gaussian marginals, with any joint law, where k is fixed, and choose one of them measurably, denoting the result by Z . Then its law λ satisfies $d\lambda/d\gamma \leq k$ and $\text{Ent}(\lambda | \gamma) \leq \log k$. For every statistic having Gaussian variance $O(N)$, its expectation under λ differs from its Gaussian expectation by $O(\sqrt{N})$; this follows from Cauchy–Schwarz and $\mathbb{E}_\gamma(d\lambda/d\gamma - 1)^2 \leq k - 1$. This applies to $|G|^2$, $|\mathcal{T}(G)|^2$ and $\log \det D\mathcal{T}(G)$. It also applies to the linear term produced by a deterministic translation of size $O(\sqrt{N})$. After choosing the scalar normalization by the actual second moment, the change-of-variables formula consequently gives

$$\frac{2}{N} \text{Ent} \left(\mathcal{L} \left((\mathcal{T}(Z) + z_N)/\sqrt{M_N} \right) \middle| \gamma \right) = \log M_N - \frac{2}{N} \mathbb{E}_\gamma \log \det D\mathcal{T}(G) + o(1),$$

where $|z_N| = O(\sqrt{N})$ and $M_N = N^{-1} \mathbb{E} |\mathcal{T}(Z) + z_N|^2$. Thus a fixed finite choice of proposals costs no leading exponential entropy. The equality includes an $o(1)$ term absorbing both the selector entropy and the change in Gaussian radial energy.

The two-proposal support bound. At exact mean latitude α , set $w = 1 - \alpha^2$ and write each orthogonal neighbor in residual coordinates as y_0 . Then

$$|y_0|^2 = c^2 = \frac{2w-1}{w}, \quad y_0 \cdot z_0 \geq -d, \quad d = \frac{1-w}{w}, \quad c^2 + d = 1.$$

Let

$$\mathcal{H}(G) = aG + V(G), \quad a = 1 - \beta\rho, \quad V = \sum_{j \geq 1} b_j R_{2j}, \quad b_j = \beta\rho^{2j} - s\sigma^{2j-2},$$

and put $Q = a^2 + \sum_j b_j^2 a_{2j}$. For Gaussian proposals with scalar correlation r , the normalized Gram matrix of their support fields converges in probability to the matrix with diagonal Q and off-diagonal

$$q_H(r) = a^2 r + \sum_{j \geq 1} b_j^2 a_{2j} r^{2j}.$$

The same holds after deleting the fixed two-dimensional latitude space. This follows from Hermite orthogonality and Gaussian Poincare; the gradients of the relevant quadratic statistics have squared expectation $O(N)$. Set $b = (Q + q_H(r))/2$ and $\ell = \max\{0, (c^2 - d)/2\}$. Choosing the better of the two proposals gives the support estimate

$$\frac{1}{\sqrt{N}} \min_{i=1,2} \sup_{y_0} \langle \mathcal{H}(G_i), y_0 \rangle \leq \max_{\ell \leq u \leq c^2} \left(\sqrt{bu} + \sqrt{(Q-b)(c^2-u)} \right) + o(1). \quad (66)$$

Indeed, choose support maximizers y, z , put $p = (y+z)/2$, $q = (y-z)/2$, and use $|p|^2 + |q|^2 = c^2$, $|p|^2 \geq \ell$. The smaller support is bounded by their average, which is at most $|\frac{1}{2}(H_1+H_2)||p| + |\frac{1}{2}(H_1-H_2)||q|$. Compactness allows suprema or limiting maximizers without a measurability difficulty.

The right side equals $c\sqrt{Q}$ unless $b/Q < \ell/c^2$. In the strict improvement regime, necessarily $w > 2/3$, it is

$$\sqrt{\frac{b(3w-2)}{2w}} + \sqrt{\frac{Q-b}{2}}.$$

Including the mean term, the squared translation cost is then

$$\left[\sqrt{\frac{b(3w-2)}{2}} + \sqrt{\frac{w(Q-b)}{2}} + \beta\sqrt{hw(1-w)} \right]^2.$$

One can obtain a moment-linear upper bound valid in both regimes. For $A \geq B > 0$, $C > 0$, it is enough, for $w \geq 2/3$, that

$$\frac{3w-2}{2A} + \frac{w}{2B} + \frac{w(1-w)}{C} \leq 1;$$

the resulting squared cost is at most $Ab + B(Q-b) + C\beta^2 h$. For $1/2 \leq w \leq 2/3$ the sufficient condition is $(2w-1)/B + w(1-w)/C \leq 1$. These are quadratic inequalities on compact intervals and are amenable to the same exact scalar-certificate machinery.

A limitation covering arbitrarily many fixed proposals. This use of pairwise acuteness cannot improve the sphere support bound at a given spectrum and latitude if

$$\min_{-1 \leq r \leq 1} q_H(r) \geq -\frac{d}{c^2} Q. \quad (67)$$

Indeed, consider any finite jointly Gaussian family whose pairwise cross-covariances have the scalar form $\mathbb{E}G_i G_j^T = r_{ij} I$. The normalized field directions $y_i = cH_i/\sqrt{NQ}$ have norm $c + o(1)$, and under strict inequality they satisfy all pairwise residual constraints $y_i \cdot y_j \geq -d + o(1)$. At the limiting Gram-matrix level these directions attain $c\sqrt{Q}$ for every proposal simultaneously. Thus the relaxation retaining only residual norms and pairwise inner products is already sharp; changing the number of proposals, their weights, or their scalar Gaussian correlations cannot help it. This obstruction makes no assertion about anisotropic cross-covariances. The equality case follows by

closure of the Gram-matrix relaxation. This is a limitation of that relaxation, not a theorem that an actual cone realizes these support directions jointly with the specified Hermite spectrum.

For the final relaxed spectrum, a numerical diagnostic gives $a^2 \simeq 0.11208784$, $Q \simeq 0.19575444$, with the dominant even-field contributions approximately 0.08022922 and 0.00343737. The minimum is near $r = -0.672487$, where $q_H(r) \simeq -0.03839184$. At $w \simeq 0.838724$, the threshold in (67) is approximately -0.04660210 . The proposed refinement is therefore inactive at that profile. These decimal values are exploratory diagnostics, not part of a rigorous numerical exponent certificate.

13.3. Several random supporting half-spaces: a Jensen obstruction. One may try to keep several cone constraints simultaneously by using the positive compensator

$$K(x) = \sum_{\ell=1}^r \beta_\ell (P_{\rho_\ell} R(x) - \rho_\ell x) \in D, \quad \beta_\ell \geq 0.$$

Write $B = \sum_\ell \beta_\ell > 0$ and $p = B^{-1} \sum_\ell \beta_\ell \rho_\ell$. The resulting support field has Gaussian coefficient $1 - Bp$, mean penalty $B\mu$, and even-chaos coefficients

$$d_j = \sum_\ell \beta_\ell \rho_\ell^{2j} - s\sigma^{2j-2}.$$

Jensen's inequality gives $\sum_\ell \beta_\ell \rho_\ell^{2j} \geq Bp^{2j}$. If

$$p^2 \geq \sigma^2, \quad Bp^2 \geq s,$$

then $Bp^{2j} - s\sigma^{2j-2} \geq 0$ for every $j \geq 1$. Replacing all the compensators by the single one $B(P_p R - pI)$ therefore preserves the Gaussian coefficient and the mean penalty and decreases every squared even-chaos coefficient. It cannot increase the scalar support bound, for any admissible spectrum. The displayed inequalities hold with considerable slack at the current optimized parameters. Positive mixtures of cone compensators consequently give no local improvement there, although this argument makes no assertion outside that parameter region.

14. LIMITATIONS AND POSSIBLE FURTHER IMPROVEMENTS

14.1. A rigorous floor for the uniform transport criterion. There is a distinction between further restrictions on the Hermite coefficients and changes to the geometric transport. The positive orthant already imposes a quantitative obstruction on the former. The following statement concerns the criterion based on the uniform latitude cost $\mathcal{M}$, with the same centered smoothed-reflection transport. It is not a lower bound for Borsuk's problem.

Proposition 14.1. *Even if the Jacobian entropy is evaluated exactly and every valid constraint on the reflection moment data is imposed, the common-parameter, uniform-latitude transport criterion when required to hold for every self-dual cone, cannot certify an exponential base below*

$$\sqrt{\frac{843}{625}} = 1.1613784912 \dots$$

Proof. Take $D = \mathbb{R}_+^N$. Then $R(G) = (|G_1|, \dots, |G_N|)$ and

$$h = \frac{2}{\pi}, \quad a_2 = \frac{1}{\pi}, \quad a_4 = \frac{1}{12\pi}, \quad a_6 = \frac{1}{40\pi}.$$

Indeed $\mathbb{E}|G_1| = \sqrt{2/\pi}$, and the second and fourth Hermite coefficients follow from the general formula below; the first two are obtained from $\mathbb{E}[|G_1|H_2(G_1)] = \sqrt{2/\pi}$ and $\mathbb{E}[|G_1|H_4(G_1)] = -\sqrt{2/\pi}$.

Write $h_0 = 7/11 < 2/\pi$, $q = \sigma^2$, $r = \rho$, $z = \zeta$, and retain only Hermite degrees two, four, and six. Put

$$c_0 = \frac{h_0}{2}, \quad c_1 = \frac{h_0}{24}, \quad c_2 = \frac{h_0}{80},$$

$$v_0 = \sum_{j=0}^2 c_j q^{2j}, \quad c'_0 = \sum_{j=0}^2 c_j q^j r^{2j+2}, \quad d_0 = \sum_{j=0}^2 c_j r^{4j+4}, \quad w_0 = \sum_{j=0}^2 (2j+2) c_j q^{2j}.$$

Because every retained term is a square, the geometric second-moment bound satisfies

$$\mathcal{M} \geq 1 + A(1 - \beta r)^2 + s^2 v_0 + A(s^2 v_0 - 2s\beta c'_0 + \beta^2 d_0) + Bh_0 \beta^2.$$

Minimizing this quadratic over all real β gives

$$\mathcal{M} \geq M_0 := (1 + A)(1 + s^2 v_0) - \frac{A^2(r + sc'_0)^2}{A(r^2 + d_0) + Bh_0}.$$

This remains a lower bound when β is constrained to be nonnegative.

The exact paired entropy cost is

$$\mathcal{D} = -\frac{1}{N} \mathbb{E} \operatorname{tr} \log(I - s^2 J_\sigma^2).$$

The scalar inequality $-\log(1 - u) \geq u$ gives $\mathcal{D} \geq s^2 W \geq s^2 w_0$. Therefore

$$\mathcal{M} e^{\mathcal{D}} \geq M_0(1 + s^2 w_0).$$

The parameters satisfy $0 \leq r, s, q \leq 1$ and $0 < z \leq 1$; allowing s and q to vary independently enlarges the actual domain $s^2 < q$.

Here is an exact polynomial certificate for the last quantity. Put $\alpha = 1 + z^2$, $d = 2(r^2 + d_0) + h_0 z$, and

$$\begin{aligned} \mathcal{P} = & [(2z + \alpha)(1 + s^2 v_0)d - 2\alpha(r + sc'_0)^2](1 + s^2 w_0) \\ & - 2\frac{843}{625}zd. \end{aligned}$$

Clearing the positive denominator shows that $M_0(1 + s^2 w_0) \geq 843/625$ is equivalent to $\mathcal{P} \geq 0$. Its boundary value at $z = 0$ is the sum of squares

$$\begin{aligned} \mathcal{B} = & 2 \left[\sum_{j=0}^2 c_j (r^{2j+2} - srq^j)^2 \right. \\ & \left. + s^2 \sum_{0 \leq j < k \leq 2} c_j c_k (q^j r^{2k+2} - q^k r^{2j+2})^2 \right] (1 + s^2 w_0). \end{aligned}$$

Thus $\mathcal{P} = \mathcal{B} + z\mathcal{R}$ for a rational polynomial $\mathcal{R}$. The accompanying standard-library program `verify_borsuk_hessian.py` partitions $[0, 1]^4$ into 310 dyadic boxes. On each box, every tensor-product Bernstein coefficient of either $\mathcal{P}$ or $\mathcal{R}$ is nonnegative. All operations use integer fractions; the subdivision has 619 nodes and maximum depth 22. Since $\mathcal{B}, z \geq 0$, this proves $\mathcal{P} \geq 0$ on the entire cube. □

The optimization over the latitude majorant is itself exact for a fixed transport. Namely, for $Q, b \geq 0$,

$$\inf_{0 < z \leq 1} \left(\frac{1 + z^2}{2z} Q + \frac{1 + z^2}{4} b^2 \right) = \max_{0 \leq t \leq 1} \left(\sqrt{tQ} + \frac{b}{2} \sqrt{1 - t^2} \right)^2.$$

For $Q, b > 0$, both sides have their extremum at the unique $z = t \in (0, 1)$ satisfying $Q(1 - z^2) = b^2 z^3$; the boundary cases follow by continuity. Thus merely replacing the optimized latitude majorant by its exact maximum does not evade the obstruction. Additional information about which latitudes and residual directions actually occur, or a different transport, can evade it. Choosing a different transport separately for each latitude is also outside the stated uniform criterion.

For context, the complete orthant spectrum is explicit:

$$a_{2j} = \frac{2}{\pi} \frac{((2j-3)!!)^2}{(2j)!} \quad (j \geq 1).$$

Equivalently, its correlation function is

$$K(t) = \mathbb{E}|G_1| |tG_1 + \sqrt{1-t^2}G'_1| = \frac{2}{\pi} (\sqrt{1-t^2} + t \arcsin t).$$

The exact smoothed one-dimensional Jacobian is

$$J_\sigma(x) = \frac{1}{\sigma} \operatorname{erf} \left(\frac{\sigma x}{\sqrt{2(1-\sigma^2)}} \right).$$

Numerical optimization using these exact expressions gives a best base approximately 1.16189803 for the same uniform criterion. This value is an exploratory numerical minimum, not the certified floor of Proposition 14.1.

14.2. Coloring by generating rays. The orthant is not a difficult example for the actual partition problem. Its orthogonality graph has an N -coloring: assign each unit vector to a coordinate where it is positive. The following robust version applies to every finitely generated acute cone.

Proposition 14.2. *Suppose an acute cone in $\mathbb{R}^N$ is generated by M unit vectors. Its unit section can be partitioned into at most M sets such that*

$$x \cdot y \geq N^{-2}$$

whenever x, y belong to the same set.

Proof. By conical Caratheodory, every unit vector x in the cone admits a representation $x = \sum_{i \in I} c_i u_i$, where $c_i \geq 0$ and $|I| \leq N$. The triangle inequality gives $\sum c_i \geq 1$, so some coefficient is at least $1/N$. Assign x to one such generator. If x and y receive the same generator, their inner product is at least the product of these two coefficients: all other pairwise generator inner products are nonnegative by acuteness. $\square$

Consequently the coloring remains valid for graphs whose edges have inner product less than N^{-2} . For example, shell and net widths N^{-4} make the edge inner products smaller than N^{-2} , while the logarithm of the net size is only $O(N \log N)$. Thus this is compatible with the thin-shell and finite-net reductions. In particular, a given common cone with $M = \exp(o(N))$ generators can be handled at subexponential cost without a Gaussian transport. For a general cone, controlling the number of required generators is a separate geometric problem; the cone obtained by completing an arbitrary fine net need not have a subexponential number. The proposition explains why a floor for a moment-based criterion need not be a genuine obstruction to Borsuk's problem.

14.3. A minimum latitude can be invisible to the moments. A restriction on all directions of a cone is stronger than a restriction on its selected vertices. Nevertheless, even that stronger restriction can be invisible to the Gaussian reflection at exponential accuracy. This explains one obstruction to extracting a universal improvement solely from the extra common latitude.

Proposition 14.3. *Let $D = D^* \subset \mathbb{R}^N$, let $R = 2P_D - I$, and put $\mu = \mathbb{E}R(G)$, $h = |\mu|^2/N$, $f = \mu/|\mu|$. Fix $\varepsilon > 0$ and*

$$0 < \alpha \leq \sqrt{h/2} - \varepsilon.$$

There exists a self-dual cone D' such that

$$D \cap \{x : f \cdot x \geq \alpha|x|\} \subset D' \subset \{x : f \cdot x \geq \alpha|x|\}.$$

Moreover, for $R' = 2P_{D'} - I$,

$$\mathbb{P}\{R'(G) \neq R(G)\} \leq 4 \exp \left[-\frac{\varepsilon^2 N}{16(1+\alpha)^2} \right].$$

The normalized squared L^2 distance between R' and R , and between their almost-everywhere Jacobians, tends to zero exponentially. In particular their mean energies and each fixed Hermite energy differ by $o(1)$ after normalization by N .

Proof. For $0 < c < 1$, denote the circular cone $\{x : f \cdot x \geq c|x|\}$ by C_c . Set $C = D \cap C_\alpha$ and $H = C_{\sqrt{1-\alpha^2}}$. Since $\alpha < 1/\sqrt{2}$, the cone H is acute. Every vector of C has nonnegative inner product with every vector of H : equivalently, $H = C_\alpha^*$. Thus the closed cone generated by $C \cup H$ is acute. A self-dual completion D' contains both C and H , and hence

$$C \subset D', \quad D' = (D')^* \subset H^* = C_\alpha.$$

Write $p = P_D G$ and $q = P_D(-G)$. Since $\mathbb{E}p = \mu/2$ and $\mathbb{E}|p|^2 = N/2$,

$$\mathbb{E}(f \cdot p) = \frac{\sqrt{hN}}{2}, \quad \mathbb{E}|p| \leq \sqrt{N/2}.$$

Both $f \cdot P_D G$ and $|P_D G|$ are 1-Lipschitz Gaussian functions. Put $t = \varepsilon/[2\sqrt{2}(1+\alpha)]$. Except on an event of probability at most $2e^{-t^2 N/2}$,

$$f \cdot p > \left(\frac{\sqrt{h}}{2} - t \right) \sqrt{N}, \quad |p| < \left(\frac{1}{\sqrt{2}} + t \right) \sqrt{N}.$$

The first right-hand side exceeds α times the second, because $\sqrt{h/2} - \alpha \geq \varepsilon$. The same assertion holds for q . The union bound proves that $p, q \in C$ with the displayed probability. On this event, $p, q \in D'$, $p \perp q$ and $G = p - q$; uniqueness of Moreau decomposition gives $P_{D'} G = p$, $P_{D'}(-G) = q$, and $R' = R$.

The event on which at least one of p, q fails its strict cap inequality is conic and therefore independent of the Gaussian radius. Since $|R'(G) - R(G)| \leq 2|G|$,

$$N^{-1} \mathbb{E}|R'(G) - R(G)|^2 \leq 4 \mathbb{P}\{p \notin \text{int } C_\alpha \text{ or } q \notin \text{int } C_\alpha\}.$$

On the open event with both cap inequalities strict, the two reflection maps agree locally. Their Jacobians therefore agree almost everywhere there; elsewhere their operator-norm difference is at most two. The same normalized bound follows for the Jacobians. Finally, orthogonal Hermite projection contracts L^2 norm, which proves the asserted energy stability. Since $|\mu| \geq \sqrt{N/2}$, the normalized axes $\mu/|\mu|$ and $\mu'/|\mu'|$ also differ exponentially. Thus the minimum latitude relative to the new cone's own mean axis is at least $\alpha - o(1)$. $\square$

Because every cone reflection has $h \geq 1/2$, a fixed cutoff $\alpha < 1/2$ can always be imposed by this construction while changing the reflection moments by $o(1)$. The mean latitude governing the numerical maximum of the current transport remains about 0.40. Consequently a restriction that all cone directions have at least this latitude, by itself, does not exclude any asymptotic moment data already admitted by a self-dual cone. This is a limitation of that proposed geometric refinement, not a lower bound for Borsuk's number. It also does not exclude using the detailed selected vertex set, its exposed faces, or constraints asserting that particular directions generate the cone.

14.4. Choosing transport parameters separately on each latitude. For a fixed $w \in (1/2, 1)$, the exact geometric support term can be bounded using a free scalar $u > 0$ by

$$\left(\sqrt{(2w-1)Q_0} + \beta \sqrt{hw(1-w)} \right)^2 \leq A_w(u)Q_0 + B_w(u)\beta^2 h,$$

where

$$A_w(u) = (2w - 1)(1 + u), \quad B_w(u) = w(1 - w)(1 + u^{-1}).$$

Thus all transport parameters may be optimized separately on each coarse mean-latitude part. This is a valid enlargement of the parameter choice, and u is simply the tangent parameter for a geometric mean. At the stationary worst latitude of the existing uniform majorizer, $t = 2w - 1 = \zeta$, the choice $u = (1 - \zeta^2)/(2\zeta^2)$ gives exactly

$$A_w(u) = \frac{1 + \zeta^2}{2\zeta}, \quad B_w(u) = \frac{1 + \zeta^2}{4}.$$

Numerical optimization with the existing Hermite relaxation found the same worst value after allowing this adaptation; this is numerical evidence of a saddle point and is not a proved global optimality statement. The exact formulas above are the rigorous part of this observation.

14.5. Which information survives at high degree? There are two different reasons for using higher moments. Scalar moments control the eigenvalue tails of the Gaussian series L . Hessian tests control the Hermite degrees in which the correction from L to the bounded field J must occur. It is essential to keep both pieces of information. In particular, the all-order top-chaos estimate (57) does not by itself control the lower chaoses of a high power of L .

A precise limitation applies to the joint polynomial tests when their Hessian projections are replaced by an unprojected variance bound. If P is odd of degree at most $2q - 1$, the proof of the joint test gives

$$(\mathbb{E}[XP(X) - |P(X)|])_+^2 \leq \left(\sum_{j=2}^q 2ja_{2j} \right) \mathbb{E}(P(X) - rX)^2 \quad (r \in \mathbb{R}).$$

Replacing the partial sum here by the full derivative budget $1 - m$ makes every such test follow already from $\mathbb{E}X \geq m$ and $\mathbb{E}X^2 = m$. Indeed, for $m > 0$, put $c = m/\mathbb{E}X$. Then $\sqrt{m} \leq c \leq 1$,

$$\mathbb{E}[(X - c)X] = 0, \quad \mathbb{E}(X - c)^2 = c^2 - m \leq 1 - m.$$

Since $|P(X)| \geq cP(X)$,

$$\mathbb{E}[XP(X) - |P(X)|] \leq \mathbb{E}[(X - c)(P(X) - rX)],$$

and Cauchy–Schwarz proves the assertion. The case $m = 0$ is immediate. Thus the useful constraint is on a *finite partial sum* of Hermite energies. This redundancy statement does not apply when the smaller fully symmetric Hessian projection norms are retained.

The fixed-scale entropy hierarchy of Section 7 also has a definite limit. At each fixed $0 < q < 1$, the ratios in (30) satisfy $1 - \kappa_r = O(r^{-2})$. Hence $K_\infty := \prod_{r=3}^\infty \kappa_r > 0$, and

$$K_\infty[-\log(1 - u) - u] \leq \mathcal{R}(u) \leq -\log(1 - u) - u.$$

Consequently this hierarchy alone retains a logarithmic divergence as $u = s^2/\sigma^2 \uparrow 1$. Increasing the truncation order can sharpen the entropy coefficient, but does not remove the need for a different estimate at that boundary. None of the numerical certificates here asserts global optimality of the chosen transport parameters, or completeness of a finite or infinite tensor relaxation.

14.6. Mixing smoothing scales. There is also a simple obstruction to improving the active two-degree transport by mixing smoothing scales. For a positive mixture $F = \int_0^1 (R_u - \mu) dc(u)$, put $t_1 = \int u^2 dc(u)$, $t_2 = \int u^4 dc(u)$, and $L_0 = \int u dc(u)$. Holder's inequality gives $t_1^3 \leq L_0^2 t_2$. A single scale $u_0 = \sqrt{t_2/t_1}$ attains equality while matching t_1, t_2 . Even a signed mixture has the same lower bound for its triangle-inequality Lipschitz estimate $L_0 = \int u d|c|(u)$ when $1/2 \leq u_0 \leq 1$:

$$\left| \frac{3u}{2u_0} - \frac{u^3}{2u_0^3} \right| \leq 1 \quad (0 \leq u \leq 1)$$

and integration against $u \, dc(u)$ gives $L_0 \geq t_1/u_0$.

It is an obstruction for this particular Lipschitz estimate at fixed first two coefficients, not an optimality theorem for Gaussian transports or for Borsuk's problem.

APPENDIX A. A SIXTH-MOMENT CONSTRAINT FROM TENSOR SYMMETRIZATION

Let $T = (T_{ijk})$ be a real fully symmetric tensor, let $(A_i)_{jk} = T_{ijk}$, and suppose $S = \sum_i A_i^2 \preceq I$. For a standard Gaussian vector $G \in \mathbb{R}^N$, put $L = \sum_i G_i A_i$ and

$$m = N^{-1} \operatorname{tr} S, \quad t_j = N^{-1} \mathbb{E} \operatorname{tr} L^j.$$

Then

$$t_4 \geq \frac{3}{2} m^2, \quad t_6 \leq \frac{17}{2} t_4 - \frac{51}{4} m^2 + \frac{9}{4} m^3. \quad (68)$$

The second estimate constrains the sixth moment in terms of the fourth moment and the first-chaos energy. The proof below uses the full symmetry of T in addition to the matrix contraction budget.

Here is a proof using only finite tensor identities and $0 \preceq S \preceq I$. Define

$$\begin{aligned} r &= \operatorname{tr} S^2, \quad r_0 = \operatorname{tr} S^3, \quad Q = \sum_{i,j} \operatorname{tr}(A_i A_j A_i A_j), \\ r_1 &= \sum_{i,j} \operatorname{tr}(S A_i A_j A_i A_j), \quad r_2 = \sum_i \operatorname{tr}(A_i S A_i S), \\ \Phi(X) &= \sum_i A_i X A_i, \quad R_0 = \sum_i \|\Phi(A_i)\|_{\text{HS}}^2, \quad K_0 = \sum_{i,j,k} \operatorname{tr}(A_i A_j A_k A_i A_j A_k). \end{aligned}$$

Set

$$B_{abcd} = \sum_i T_{iab} T_{icd}, \quad H_{abcd} = B_{abcd} + B_{acbd} + B_{adbc}.$$

Wick's formula and contraction of indices give

$$N t_4 = 2r + Q, \quad \|H\|^2 = 3r + 6Q, \quad N t_6 = 2r_0 + 6r_1 + 3r_2 + 3R_0 + K_0. \quad (69)$$

The fourth-moment estimate follows at once.

Two pieces of information should be retained when estimating the last expression in (69). First, the five-index chain tensor

$$U_{ab|c|de} = \sum_{p,q} T_{abp} T_{pcq} T_{qde}$$

gives the following sum-of-squares identity. Define

$$W_{abcde} = \sum_{\text{cyc}(a,b,c)} (U_{ad|c|be} + U_{ae|c|bd} - 2U_{ab|c|de}).$$

If the three cyclic summands are V_1, V_2, V_3 , contraction gives

$$\|V_i\|^2 = 6(r_2 - K_0), \quad \langle V_i, V_j \rangle = 6(r_1 - R_0) \quad (i \neq j).$$

Hence

$$\|W\|^2 = 18(r_2 + 2r_1 - 2R_0 - K_0).$$

Consequently

$$K_0 \leq r_2 + 2r_1 - 2R_0. \quad (70)$$

This is the new contraction inequality; the weaker estimate $K_0 \leq r_2$ loses a substantial amount of information.

Second, the weighted norm of the symmetric four-index tensor satisfies

$$\|H\|_S^2 := \sum_{a,a',b,c,d} S_{aa'} H_{abcd} H_{a'bcd} = 3r_2 + 6r_1 \leq \|H\|^2. \quad (71)$$

The three diagonal products in this expansion are r_2 , and each of the six cross products is r_1 . The inequality uses $S \preceq I$.

To finish, put

$$C_{ijk} = \text{tr}(A_i A_j A_k), \quad M_{abc} = \frac{1}{3} \sum_d (S_{ad} T_{dbc} + S_{bd} T_{adc} + S_{cd} T_{abd}).$$

Direct contraction gives

$$\|C\|^2 = R_0, \quad \langle C, M \rangle = r_1, \quad \|M\|^2 = (r_0 + 2r_2)/3.$$

The map $T \mapsto B$ has derivative of Frobenius operator norm at most 2, because the flattening $T_{i,(a,b)}$ has operator norm at most one. The map $B \mapsto H$ has operator norm at most three. Moreover, $\nabla \|H\|^2 = 12M + 24C$ within the fully symmetric tensor space. It follows that

$$\|12M + 24C\| \leq 12\|H\|,$$

or equivalently

$$R_0 + r_1 + \frac{r_0 + 2r_2}{12} \leq \frac{1}{4}\|H\|^2. \quad (72)$$

Normalize the contractions by writing

$$v = r/N, \quad w = r_0/N, \quad u = r_2/N, \quad z = r_1/N, \quad d = R_0/N, \quad \delta = t_4 - \frac{3}{2}v.$$

Then $\delta \geq 0$, $u \leq w$, and (71)–(72) become

$$z \leq \delta - u/2, \quad d + z + (w + 2u)/12 \leq \frac{3}{2}\delta.$$

The inequality $u \leq w$ follows from $2(r_0 - r_2) = \sum_i \|[A_i, S]\|_{\text{HS}}^2$. Now (69) and (70) give

$$\begin{aligned} t_6 &\leq 2w + 8z + 4u + d \\ &\leq \frac{23}{12}w + \frac{23}{6}u + 7z + \frac{3}{2}\delta \\ &\leq \frac{23}{12}w + \frac{1}{3}u + \frac{17}{2}\delta \leq \frac{9}{4}w + \frac{17}{2}\delta. \end{aligned}$$

Finally, the eigenvalues of S lie in $[0, 1]$. Thus

$$m^2 \leq v \leq m, \quad (m - v)^2 \leq (v - w)(1 - m).$$

For $m < 1$,

$$t_6 \leq \frac{17}{2}t_4 - \frac{51}{4}v + \frac{9}{4}w \leq \frac{17}{2}t_4 - \frac{21}{2}v - \frac{9}{4} \frac{(m - v)^2}{1 - m}.$$

The last expression decreases for $v \geq m^2$, since its derivative is at most $-21/2 + (9/2)m < 0$. Substituting $v = m^2$ proves (68). The cases $m = 0, 1$ are immediate.

REFERENCES

- [1] K. Borsuk, *Drei Sätze über die n -dimensionale euklidische Sphäre*, Fundamenta Mathematicae 20 (1933), 177–190. [DOI](#).
- [2] J. Kahn and G. Kalai, *A counterexample to Borsuk's conjecture*, Bulletin of the American Mathematical Society 29 (1993), 60–62. [arXiv](#).
- [3] O. Schramm, *Illuminating sets of constant width*, Mathematika 35 (1988), 180–189. [DOI](#).
- [4] J. Bourgain and J. Lindenstrauss, *On covering a set in $\mathbb{R}^N$ by balls of the same diameter*, in Geometric Aspects of Functional Analysis (1989–90), Lecture Notes in Mathematics 1469, Springer, 1991, 138–144. [DOI](#).
- [5] G. P. Barker and J. Foran, *Self-dual cones in Euclidean spaces*, Linear Algebra and its Applications 13 (1976), 147–155. [DOI](#).
- [6] J. J. Moreau, *Décomposition orthogonale d'un espace hilbertien selon deux cônes mutuellement polaires*, Comptes Rendus de l'Académie des Sciences de Paris 255 (1962), 238–240.
- [7] H. Federer, *Geometric Measure Theory*, Grundlehren der mathematischen Wissenschaften 153, Springer, 1969.
- [8] D. Bakry, I. Gentil and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Grundlehren der mathematischen Wissenschaften 348, Springer, 2014. [DOI](#).

- [9] S. Janson, *Gaussian Hilbert Spaces*, Cambridge Tracts in Mathematics 129, Cambridge University Press, 1997.
- [10] G. Kalai, *Some old and new problems in combinatorial geometry I: around Borsuk's problem*, in *Surveys in Combinatorics 2015*, London Mathematical Society Lecture Note Series 424, Cambridge University Press, 2015, 147–174. [arXiv](#).
- [11] *An exponential improvement for Borsuk's problem*, AI-originated mathematical notes, curated by Andriy Prymak, version 2610-01v1, October 6, 2026. <https://prymak.net/aiomn/2610-01v1.pdf>.
- [12] A. Hall, A. Prymak and C. Sujsuntinukul, *Exponential bounds for the spherical Blaschke–Lebesgue problem*, 2026. [arXiv:2606.30960](#).
- [13] J. A. Tropp, *An introduction to matrix concentration inequalities*, Foundations and Trends in Machine Learning 8 (2015), 1–230. [Author's version](#).