

TRANSLATIVE COVERING VIA ADDITIVE COMBINATORICS AND DISCRETIZATION

ABSTRACT. An elementary argument from additive combinatorics shows that a compact set $A \subset \mathbb{R}^n$ can be covered by at most

$$(n \log n + n \log \log n + O(n)) \frac{\min\{|A+L|, |A-L|\}}{|L|}$$

translates of a convex body L . This extends the asymptotic Rogers–Zong estimate, which contains only the term $|A-L|$. The proof combines a weighted version of an estimate recorded by Tao and Vu with the harmonic estimate for greedy covering. A fine lattice supplies a finite additive problem, while a maximal separated subset controls the number of target points in each translate. All ingredients of the proof are included. Applications recover the recent $2^{n+o(n)}$ upper bound for the illumination problem and give a $3^{n+o(n)}$ upper bound for the covering index of arbitrary convex bodies.

1. THE COVERING ESTIMATE

A convex body is a compact convex subset of $\mathbb{R}^n$ with nonempty interior. For a compact set A and a convex body L , let $N(A, L)$ be the least number of translates of L whose union contains A . Write $|E|$ for Lebesgue volume, $\#F$ for the cardinality of a finite set, and $A \pm L = \{a \pm \ell : a \in A, \ell \in L\}$. All logarithms are natural.

The Rogers–Zong estimate [7], combined with Rogers’ bound for translative covering density [6], gives, for convex bodies A, L ,

$$N(A, L) \leq (n \log n + n \log \log n + O(n)) \frac{|A-L|}{|L|}.$$

The following theorem extends this asymptotic estimate by replacing $|A-L|$ with $\min\{|A+L|, |A-L|\}$.

Theorem 1.1. *Let $A \subset \mathbb{R}^n$ be nonempty and compact, and let $L \subset \mathbb{R}^n$ be a convex body. For every $0 < \delta < 1$,*

$$N(A, L) \leq \frac{1 + n \log(4/\delta - 2)}{(1 - \delta)^n} \frac{\min\{|A+L|, |A-L|\}}{|L|}. \quad (1.1)$$

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Mathematical context. Developed in response to Gorodzha’s work on fractional illumination and the optimal exponential rate in Hadwiger’s covering conjecture [2], and an inquiry into a more elementary approach and earlier references for the proof ingredients. Sources of these ingredients are credited in the text.

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Consequently, as $n \rightarrow \infty$, uniformly over A and L ,

$$N(A, L) \leq (n \log n + n \log \log n + O(n)) \frac{\min\{|A + L|, |A - L|\}}{|L|}. \quad (1.2)$$

The implicit constant is absolute.

For explicit statements below, when $n \geq 2$ set

$$\delta_n = \frac{1}{n \log n}, \quad C_n = \frac{1 + n \log(4n \log n - 2)}{(1 - 1/(n \log n))^n}. \quad (1.3)$$

Then the coefficient in (1.2) may be taken to be C_n , and

$$C_n = n \log n + n \log \log n + (1 + \log 4)n + O\left(\frac{n \log \log n}{\log n}\right). \quad (1.4)$$

Idea of the proof. The discrete tool is a weighted version of the estimate from additive combinatorics in Tao–Vu [10, Corollary 2.10, p. 63]. It says that some translate captures a prescribed proportion of any weights placed on the target set. Applying this observation to the weights assigned by greedy covering gives a logarithmic factor depending only on the largest number of target points in one translate. To exploit this local dependence, a fine lattice is used to discretize the problem, and a maximal separated subset of its points near A is retained. A packing argument bounds the local number of retained points by $(4/\delta - 2)^n$. The auxiliary lattice mesh can then tend to zero without increasing the logarithmic factor.

This proves (1.2) using finite sums, Cauchy–Schwarz, an elementary averaging argument, and volume comparisons.

Gorodzha [2] recently proved the illumination bound obtained below. The purpose of this note is to give a self-contained proof of the more general translative estimate using additive combinatorics. The illumination consequence is thus a recovery of that result. The covering-index consequence improves the general estimate in Bezdek–Khan [1, Corollary 7.3].

2. THE DISCRETE TOOL FROM ADDITIVE COMBINATORICS

The discrete estimate follows from a weighted counting argument and an elementary greedy argument. For finite subsets U, V of an abelian group, $N(V, U)$ denotes the least number of translates of U , with translations in that group, covering V . Put $H_0 = 0$ and, for $q \in \mathbb{N}$, $H_q = \sum_{j=1}^q 1/j \leq 1 + \log q$.

Lemma 2.1 (A weighted estimate from additive combinatorics). *Let U, V be finite nonempty subsets of an abelian group, and give the points of V nonnegative weights w_v of total weight $W > 0$. Set*

$$m = \#U, \quad s = \min\{\#(V + U), \#(V - U)\}.$$

Some translate $x + U$ contains weight at least mW/s :

$$\max_x \sum_{v \in V \cap (x+U)} w_v \geq \frac{mW}{s}. \quad (2.1)$$

Proof. Define

$$f_+(x) = \sum_{v \in V} w_v \mathbf{1}_U(x - v), \quad f_-(x) = \sum_{v \in V} w_v \mathbf{1}_{-U}(x - v).$$

Both functions have total sum mW , and their supports lie in $V + U$ and $V - U$, respectively. Their squared sums are equal. Indeed, if $r_U(t) = \#(U \cap (U + t))$, then $r_U(t) = r_U(-t)$, and expansion gives

$$\sum_x f_+(x)^2 = \sum_{v, v' \in V} w_v w_{v'} r_U(v - v') = \sum_x f_-(x)^2 =: E. \quad (2.2)$$

Apply Cauchy–Schwarz to the function with the smaller of the two support bounds. On the other hand, bound one factor of $f_-(x)^2$ by its maximum. This yields

$$\frac{m^2 W^2}{s} \leq E \leq (\max_x f_-(x)) \sum_x f_-(x) = mW \max_x f_-(x).$$

Finally, $f_-(x) = \sum_{v \in V \cap (x+U)} w_v$. $\square$

For weights $w_v = 1$, this is the usual estimate for the intersection with a suitable translate from additive combinatorics; see [10, Corollary 2.10]. The weighted extension is included because it is precisely what the next argument needs.

Lemma 2.2 (Greedy covering and local cardinality). *Let X be a finite nonempty set and let $\mathcal{F}$ be a family of subsets covering X , each of cardinality at most q . Suppose $F > 0$ and that, for every choice of nonnegative weights on X of total weight $W > 0$, some member of $\mathcal{F}$ has weight at least W/F . Then X can be covered by at most FH_q members of $\mathcal{F}$.*

Proof. At each step choose a member of $\mathcal{F}$ containing the largest number k of uncovered points, and assign weight $1/k$ to each newly covered point. There are finitely many distinct subsets of X , so this choice exists. If the algorithm takes M steps, the total weight is M .

Fix $E \in \mathcal{F}$, and order its $r \leq q$ points by the time when they are covered, breaking ties arbitrarily. When its j th point is covered, at least $r - j + 1$ points of E were uncovered at the start of that step. The greedy choice therefore covers at least $r - j + 1$ new points. The weight of this point is at most $1/(r - j + 1)$, so the total weight on E is at most $H_r \leq H_q$.

Apply the hypothesis to these weights. Some member has weight at least M/F , while every member has weight at most H_q . Thus $M \leq FH_q$. $\square$

This is the harmonic weight estimate underlying the Lovász–Stein covering estimate [4, 8].

Proposition 2.3 (A discrete covering bound). *For finite nonempty subsets U, V of an abelian group, put*

$$q = \max_x \#(V \cap (x + U)).$$

Then

$$N(V, U) \leq H_q \frac{\min\{\#(V + U), \#(V - U)\}}{\#U}. \quad (2.3)$$

In particular, one may replace H_q by $1 + \log \min\{\#U, \#V\}$.

Proof. Use the family $\{V \cap (x + U) : x \in V - U\}$ in Lemma 2.2. Lemma 2.1 supplies its hypothesis with $F = \min\{\#(V + U), \#(V - U)\}/\#U$. $\square$

For example, if $\#(U + V) \leq K\#U$, then $N(V, U) \leq K(1 + \log \min\{\#U, \#V\})$. The familiar formulation with $O(K \log \#V)$, for $\#V \geq 2$, is also recorded in Hanson–Petridis [3, Lemma 5]. The essential point for the geometric application is to retain q in (2.3): it can be much smaller than either of the two sets.

3. LATTICE DISCRETIZATION WITH A SEPARATED SUBSET

Lemma 3.1 (A symmetric subset; Stein [9]). *For every convex body $L \subset \mathbb{R}^n$, there is a translation L' of L such that $S = L' \cap (-L')$ is an origin-symmetric convex body satisfying*

$$|S| \geq 2^{-n}|L|. \quad (3.1)$$

Proof. Fubini’s theorem gives

$$\int_{\mathbb{R}^n} |L \cap (z - L)| dz = |L|^2.$$

The integrand vanishes outside $L + L = 2L$, whose volume is $2^n|L|$. For some z , therefore, $|L \cap (z - L)| \geq 2^{-n}|L|$. Put $L' = L - z/2$. Then $S = L' \cap (-L') = (L \cap (z - L)) - z/2$. It is convex, symmetric and of positive volume, so it has nonempty interior and contains the origin in its interior. $\square$

Milman and Pajor [5, Corollary 3] later established the stronger statement that (3.1) holds for $L' = L - g(L)$, where $g(L)$ is the centroid of L . Thus the translation can be specified in advance. The elementary averaging argument above is sufficient for the present application.

Proof of Theorem 1.1. Translate L as in Lemma 3.1; this changes neither the covering number nor the volumes in the statement. Fix $0 < \delta < 1$, and choose

$$0 < \varepsilon < \min\{\delta, (1 - \delta)/2\}.$$

There are two scales: δ controls the separation of the retained points, while ε controls the error of the lattice approximation and will tend to zero.

Step 1: form a finite lattice problem. Choose $h > 0$ so small that the cell

$$Q = [-h/2, h/2]^n \subset \varepsilon S, \quad \Gamma = h\mathbb{Z}^n.$$

Define the finite nonempty set

$$B = \{z \in \Gamma : (z + Q) \cap A \neq \emptyset\}.$$

Since the cells cover $\mathbb{R}^n$ and $Q = -Q$,

$$A \subset B + Q, \quad B \subset A + Q. \quad (3.2)$$

Choose a maximal subset $\Lambda \subset B$ such that the translates $\lambda + (\delta/2)S$, $\lambda \in \Lambda$, have pairwise disjoint interiors. Maximality and $S = -S$ imply

$$B \subset \Lambda + \delta S, \quad A \subset \Lambda + (\delta + \varepsilon)S. \quad (3.3)$$

Indeed, a point of B outside $\Lambda + \delta S$ could be added to the packing. Now put

$$T = (1 - \delta - \varepsilon)L, \quad U = \Gamma \cap T.$$

The origin belongs to U , so U is nonempty. Since $S \subset L$ and L is convex,

$$U + (\delta + \varepsilon)S \subset L. \quad (3.4)$$

Thus a cover of Λ by translates of U gives a cover of A by the same number of translates of L .

Step 2: compare cardinalities with volume. For any finite $D \subset \Gamma$, the interiors of the cells $d + Q$ are disjoint, and hence

$$\#D h^n = |D + Q|. \quad (3.5)$$

Because $\Lambda \subset A + Q$, $Q = -Q$ and $Q \subset \varepsilon S$,

$$(\Lambda + U) + Q \subset A + T + 2Q \subset A + L,$$

and likewise

$$(\Lambda - U) + Q \subset A - T + 2Q \subset A - L.$$

For these inclusions, use $T + 2\varepsilon S \subset (1 - \delta + \varepsilon)L \subset L$. It follows that

$$\#(\Lambda \pm U) h^n \leq |A \pm L|. \quad (3.6)$$

In the other direction,

$$(1 - \delta - 2\varepsilon)L \subset U + Q. \quad (3.7)$$

To see this, choose, for x in the set on the left, a lattice point u such that $x \in u + Q$. Then

$$u \in (1 - \delta - 2\varepsilon)L + Q \subset (1 - \delta - \varepsilon)L = T,$$

so $u \in U$. Consequently,

$$\#U h^n \geq (1 - \delta - 2\varepsilon)^n |L|. \quad (3.8)$$

Step 3: bound the local number of points. For any $x \in \Gamma$, the disjoint packed translates with centers in $\Lambda \cap (x + U)$ all lie in

$$x + T + (\delta/2)S \subset x + (1 - \delta/2 - \varepsilon)L.$$

Their volumes and (3.1) therefore give

$$\begin{aligned} q := \max_{x \in \Gamma} \#(\Lambda \cap (x + U)) &\leq \frac{(1 - \delta/2 - \varepsilon)^n |L|}{(\delta/2)^n |S|} \\ &\leq (4/\delta - 2)^n. \end{aligned} \tag{3.9}$$

This bound is independent of h and ε .

Step 4: apply the discrete bound. Apply Proposition 2.3 in the additive group Γ , with target Λ and covering set U . Equations (3.4)–(3.9) imply

$$N(A, L) \leq \frac{1 + n \log(4/\delta - 2)}{(1 - \delta - 2\varepsilon)^n} \frac{\min\{|A + L|, |A - L|\}}{|L|}.$$

This holds for every sufficiently small positive ε . Letting $\varepsilon \downarrow 0$ proves (1.1). There is no need to take a limit of the individual covers: the fixed integer $N(A, L)$ satisfies each of the displayed bounds.

Finally take $\delta = \delta_n$ from (1.3). As $n \rightarrow \infty$,

$$(1 - \delta_n)^{-n} = 1 + \frac{1}{\log n} + O\left(\frac{1}{(\log n)^2}\right),$$

whereas

$$1 + n \log(4n \log n - 2) = n \log n + n \log \log n + n \log 4 + O(1).$$

Multiplication proves (1.4) and (1.2). $\square$

Remark 3.2. The separation scale is $1/(n \log n)$ in the norm with unit ball S . A fixed choice $\delta = c/n$ also gives an $O(n \log n)$ factor, but with leading coefficient e^c . The logarithmic refinement retains leading coefficient 1. The much finer auxiliary lattice has no prescribed Euclidean step size: its cardinality disappears from the estimate because the greedy argument uses the local quantity q .

4. HOMOTHETIC COVERING, ILLUMINATION, AND COVERING INDEX

Corollary 4.1. *For every convex body $K \subset \mathbb{R}^n$, $n \geq 2$, and every $\lambda > 0$,*

$$N(K, \lambda K) \leq C_n(1 + \lambda^{-1})^n. \tag{4.1}$$

Proof. Apply Theorem 1.1 with $A = K$ and $L = \lambda K$, using $K + \lambda K = (1 + \lambda)K$. $\square$

Let $H(K)$ be the least number of smaller positive homothets of K covering K , and let $H(n)$ be its supremum over convex bodies in $\mathbb{R}^n$. By the classical equivalence of homothetic covering and illumination, $H(K)$ is also the illumination number of K .

Corollary 4.2 (The Hadwiger asymptotic; cf. Gorodzha [2]). *For every convex body $K \subset \mathbb{R}^n$, $n \geq 2$,*

$$H(K) \leq 2^n C_n = 2^n(n \log n + n \log \log n + O(n)). \tag{4.2}$$

In particular, $H(n) = 2^{n+O(\log n)}$. These statements apply equally to the illumination problem.

Proof. For each $0 < \lambda < 1$, a cover counted by $N(K, \lambda K)$ is a cover by smaller homothets. Thus (4.1), followed by $\lambda \uparrow 1$, gives $H(K) \leq 2^n C_n$.

A smaller homothet of $[-1, 1]^n$ contains at most one of its 2^n vertices, giving the matching exponential lower bound $H(n) \geq 2^n$. $\square$

Following Bezdek and Khan [1], put $\gamma_m(K) = \inf\{\lambda > 0 : N(K, \lambda K) \leq m\}$ and define the covering index by

$$\text{coin}(K) = \inf \left\{ \frac{m}{1 - \gamma_m(K)} : m \in \mathbb{N}, \gamma_m(K) \leq \frac{1}{2} \right\}. \tag{4.3}$$

Corollary 4.3. *For every convex body $K \subset \mathbb{R}^n$, $n \geq 2$,*

$$\text{coin}(K) \leq 2N(K, \tfrac{1}{2}K) \leq 2 \cdot 3^n C_n = 2 \cdot 3^n (n \log n + n \log \log n + O(n)). \quad (4.4)$$

Proof. For $m = N(K, \tfrac{1}{2}K)$, the inequality $\gamma_m(K) \leq 1/2$ gives $\text{coin}(K) \leq m/(1 - \gamma_m(K)) \leq 2m$. Now take $\lambda = 1/2$ in (4.1). $\square$

For arbitrary convex bodies, [1, Corollary 7.3] gives an upper bound of order $7^n \sqrt{n} \log n$. Thus (4.4) improves the exponential base in that general estimate to 3. Their estimate for centrally symmetric bodies already has exponential base 3. No optimality of the base 3 for the covering index is asserted here.

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